Diversification in Illiquid Market

Sergei Isaenko*†

Concordia University

Abstract

We consider a portfolio optimization problem for an investor who can trade liquid and illiquid stocks. The illiquidity of a stock is defined by the presence of convex transaction costs charged for its trading. We analyze the effects of the presence of illiquid stocks on allocations to liquid ones and vise versa. We find that allocations to the two types of stocks are considerably different from those in a liquid market. The differences remain significant even when illiquid stocks take a very small fraction of the whole market. Moreover, we confirm the limits to arbitrage by showing that an investor takes a risky position in the presence of arbitrage opportunities.

Keywords: Portfolio Choice; Liquidity; Transaction Costs
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*The John Molson School of Business, Concordia University, 1455 de Maisonneuve Blvd. W. Montreal, Quebec, H3G 1M8. Tel: 1-514-848-2424, ext. 2797. E-mail: sisaenko@jmsb.concordia.ca.

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1 Introduction

It has been well–known that the presence of illiquid risky securities in the market has significant effects on allocations by investors to liquid risky securities. For example, during extreme market episodes when a lot of risky securities become much less liquid, many investors try to reduce their holdings of these securities in favor of liquid ones. Interdependence between allocations to liquid and illiquid risky securities can also shed the light on the contagion and comovement of prices in illiquid markets. Despite the obvious benefits, a complete understanding of this interdependence is still lacking. This understanding relies on a realistic modelling of illiquid securities and the role of stock diversification in illiquid market. Intuitively, illiquidity of a stock assumes its limited availability for immediate trading. Because trading of an investor is mediated by a market maker who provides liquidity in the market, stock illiquidity first affects the positions of an intermediary. To avoid default, a market maker increases the bid–ask spread for an illiquid stock making its trading more costly for an investor. Because it is too expensive to trade an illiquid stock, an investor will not have the desired allocations to this stock.

The goal of this paper is to understand the role of diversification in the presence of illiquid stocks and, therefore, the influence of the allocations by an investor to illiquid stocks on those to liquid stocks and vise versa. We model stock illiquidity for an investor by assuming that transaction costs on trading this stock are convex in the rate of trading. The transaction costs include the bid–ask spread and increase with it as an investor tries to buy more shares within a given time–period. We assume that stock returns have infinite first–order variation so that an investor must trade arbitrarily fast to have his best allocations. However, the faster an investor trades the higher the costs are. As a result, a market becomes illiquid for an investor who endogenously chooses to trade at a finite rate and is not able to reach his best allocations.

The described model of an illiquid stock was introduced in Cetin and Rogers (2007), and Isaenko (2005).[1] This approach is related to the idea of Longstaff (2001) that holdings of illiquid stock by an investor should have finite first order variation. We extend this idea and connect it to the market microstructure. The other existing approaches to model stock illiquidity are considerably different and mentioned at the end of the Introduction.

Transaction costs may have linear and convex terms in the rate of trading. The linear term generates a no–trading zone. Because the effects resulting from this term have been thoroughly studied,[2] our analysis assumes that the linear component in the transaction costs is absent.

A typical investor often has to choose between large numbers of liquid and illiquid stocks.

[1] Also see Rogers and Singh (2006).

[2] The literature on portfolio optimization in the presence of proportional transaction costs is vast. Its well–known examples include Constantinides (1986), Davis and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991), Vayanos (1998, 2003), Liu and Loewestein (2002), Liu (2004), and many others.

He diversifies his portfolio, and stocks of each type are essential for removing idiosyncratic risk. As a result, an investor holds many risky securities of both types that we describe by representative stocks that carry only systematic risk. The number of representative stocks depends on the number of systematic shocks. Therefore, we consider the market where individual liquid stocks are aggregated into $n-1$ representative liquid stocks that along with an illiquid representative stock are propelled by n systematic shocks. An investor trades both types of stocks for further diversification. The total number of stocks, n, allows us to control the significance of the illiquid stock for the market. Moreover, for the purpose of tractability we assume that transaction costs are very small for liquid stocks and could be neglected.

In the special case, the risk in the illiquid representative stock could be perfectly replicated by liquid representative stocks. Therefore, arbitrage opportunities could be available to an investor. These opportunities will exist in the long run because an investor can take only limited advantage of them. Thus, it is important to understand the optimal behavior of an investor in the presence of arbitrage opportunities. For this, we analyze optimal trading of an investor in the presence of perfect correlation between an illiquid stock and a liquid stock that have different conditional Sharpe ratios. This analysis is similar to that of an arbitrage strategy that involves trading liquid and illiquid stocks of individual companies.

Assuming that the investment opportunity set is constant we arrive at the following conclusions:

1. If an investor trades one illiquid and $n-1$ liquid stocks, then an illiquid stock is likely to play a significant role in the portfolio diversification even when illiquid stocks take a very small fraction of the whole market (n is large). In particular, even when n is large, an investor often tries to reduce his holdings of the illiquid stock only by a little so his flight to liquidity is small. For example, we show that if an illiquid stock is very expensive to trade for one year and takes only 4% of the whole market, then an investor tries to reduce his holdings of this stock only by 20% from those in the liquid market. In addition, the allocations to all liquid stocks could easily be very different from those in the liquid market and an investor could substantially discount his holdings of the illiquid stock. The role of an illiquid stock in diversification becomes negligible if n is large and this stock remains illiquid for a very long time. In this paper we assume that only one of these two conditions can hold.

These results suggest that, given that the proportion of illiquid stocks in the market is noticeable, a lot of intuition about allocations between liquid and illiquid stocks can be learned from the analysis of the market with $n = 2$. This justifies our primarily emphasis on the market with two stocks. The remaining conclusions are derived from this market and are easily extended to illiquid markets with multiple liquid stocks.

2. If an investor trades two stocks one of which is illiquid, then holdings of both risky securities would be substantially different from those where both stocks are liquid. If the illiquid stock holdings are small, then an investor always sets his allocation in the liquid stock

at that in the liquid market with one stock. If the volatility of the illiquid stock holdings is too high, he adjusts allocations in the liquid stock to reduce cumulative volatility. In particular, if a correlation between the two stocks is positive, then an investor can take a short position in the liquid stock even if this position is significantly positive in a perfectly liquid market. If the two stocks are independent, then allocations to the liquid stock are similar to the case with a positive correlation but less pronounced. If the correlation is negative, then the mutual dynamics of the allocations to the two stocks is defined by the size of the correlation. Increasing interdependence between the allocations to the two stocks will cause contagion between the stocks and comovements of the two stock prices.

If the correlation between the two stocks is substantial, we find only minor differences in the allocations to the liquid stock between the cases where illiquidity is very strong, moderate or small. It follows that the described impact on the liquid stock portfolio allocations can be significant even when the other stock's illiquidity is minor, leading to incorrect conclusions about investors' rationality and market efficiency, if illiquidity is neglected.

3. In the special case where the two traded stocks have perfect correlation, we uncover the limits to arbitrage: investor considers arbitrage opportunities on an equal basis with other investment opportunities.[3] He offsets the risk of his position in the illiquid stock with the one in the liquid stock and takes an additional position in the liquid stock that is the same as the one in the absence of the illiquid stock.

4. In agreement with conclusion 1, an investor achieves a highest expected utility when his allocations are very close to those in a liquid market. This result implies that in many situations an investor barely tries to reduce his holdings of the illiquid stock in the presence of a liquid one. Thus, flight to liquidity resulting from a stock illiquidity *per se* is negligible. For a rational investor, it could occur only for very volatile stock returns given that the stock is extremely illiquid for a very long time. Otherwise, an investor has to be irrationally pessimistic in his views on the future returns of an illiquid stock. Finally, the welfare loss of an investor at allocations maximizing the expected utility is negligibly small. This result demonstrates the importance of the allocations away from those maximizing the expected utility to be analyzed on the optimal trading and welfare losses. In fact, the liquidity premium at these allocations can easily be of the same order of magnitude as the risk premium.

The related paper by Isaenko (2005) studies the optimal trading of an investor and a liquidity premium when trading the whole stock market obliges paying convex transaction costs while bond market is liquid. Isaenko finds that, with the calibration of the model on the historical data, the flight to liquidity observed for an illiquid stock market is unlikely to result from this stock illiquidity per se. It follows from either the increased volatility of stock returns or irrationally pessimistic expectations of investors. He also finds that welfare losses of an investor at allocations maximizing his utility are vary small. These conclusions

[3]Related results can be found in Basak and Croitoru (2000, 2003), Liu and Longstaff (2004), and Isaenko (2004).

are barely affected by the presence of stock crashes. Because an investor is always away from his best allocations, Isaenko studies optimal trading far from those allocations. He finds that liquidity premium at those allocations could be of the same size as a risk premium and the presence of stock crashes makes this conclusion only stronger. In the case where the convex transaction costs have linear term, Isaenko investigates the resulting no–trading zone. He shows numerically that the cross–section of this zone at fixed stock price has a shape of a connected cone which opening angle increases with smaller stock price. In addition, the boundaries of this cone change very weakly with the convexity of the transaction costs which warrants the convenience of the analysis of this zone due to the absence of singularity of stochastic control. In this paper we extend some of the above results to the case where illiquid stocks occupy only a fraction of the market (see conclusions 1 and 4). More importantly, as motivated in the beginning of the Introduction, we study the role of stock diversification for risk management in the illiquid market, interdependence of allocations between liquid and illiquid stocks as well as trading in the presence of arbitrage opportunities.

Our analysis could also be related to the paper by Kahl, Liu, and Longstaff (2003). These authors consider a portfolio choice by an entrepreneur who cannot trade his firm's stock shares but is not restricted in trading other securities. The inability of an investor to trade his firm's stock shares can be formally interpreted as though his firm's stock is illiquid. If so, our special case where the cost of trading an illiquid stock is very high resembles their model and some of their conclusions are similar to ours. However, illiquidity in our model is not extreme as investors can trade the security at a flexible and possibly high rate. Moreover, the investor in our model trades an illiquid stock that is essential for the stock market rather than a small stock with idiosyncratic risk as in Kahl, Liu, and Longstaff (2003). Overall, our analysis applies to trading by an investor in an illiquid market, rather than to an insider's transactions in a perfectly liquid market.

Our paper contributes to growing theoretical literature on the market's illiquidity. In addition to the above–cited references, important contributions include Lipman and McCall (1986), Amihud and Mendelson (1986), Grossman and Miller (1988), Grossman and Laroque (1990), Boudoukh and Whitelaw (1993), Frey (1998), Holmstrom and Tirole (2001), Weill (2002), Acharya and Pedersen (2003), Huang (2003), Duffie, Garleanu, and Pedersen (2003a, 2003b), O'Hara (2003), Vayanos and Wang (2003), Cetin, Jarrow, and Protter (2004), Cetin, Jarrow, Protter, and Warachka (2004), Eisfeldt (2004), Jang, Koo, Liu, and Loewenstein (2005), Lo, Mamaysky, and Wang (2004), Brunnermeir and Pederson (2005), He and Mamaysky (2005), Huang and Wang (2005), Liu and Yong (2005), Longstaff (2005), Pereira and Zhang (2005), Deuskar (2006), and many others.

The rest of the article is organized as follows. Section 2 describes the model for the economy with two stocks. Section 3 presents the solutions to the model. Section 4 describes trading of an investor in the presence of $n-1$ liquid stocks, where $n > 2$. Section 5 summarizes. Appendix A presents the HJB equation of the optimization problem introduced in Section 2, while Appendix B identifies the conditions for the absence of arbitrage opportunities in the market with two stocks.

2 The Basic Model

Due to easy tractability, we first analyze an optimal trading of an investor in an economy with one liquid and one illiquid stock. We extend this economy to that with multiple liquid stocks in Section 4 and show that the majority of our conclusions remains intact.

2.1 The Asset Market

We consider a markovian economy with a finite horizon T where there is a single perishable consumption good that we treat as the numeraire. We assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$. Uncertainty in the model is generated by a standard two–dimensional Brownian motion $W = (W_1, W_2)$, where W is adapted.

Investors can continuously trade three securities: a riskless bond and two stocks. The riskless bond has the price dynamics

$$dB_t = B_t r dt, \tag{1}$$

where r denotes a constant interest rate. We assume that the dynamics of the stock prices are given by

$$\begin{aligned} dS_{1t} &= S_{1t}(\mu_1 dt + \sigma_1 dW_{1t}), & (2) \\ dS_{2t} &= S_{2t}(\mu_2 dt + \sigma_2 dW_{2t}), & (3) \end{aligned}$$

where W_1 and W_2 have the constant instantaneous correlation coefficient ρ and μ_1, μ_2, σ_1, and σ_2 are also constants, the last two of which are positive.

We assume that stock 1 is illiquid so that its trading requires paying transaction costs, while stock 2 is liquid meaning its trading is free. Trading shares of stock 1 takes place at a finite rate u_1, that is

$$dN_{1t} = u_{1t} dt, \tag{4}$$

where N_1 is a number of shares of stock 1 held by an investor.

The last assumption results from the shape of the charges levied on trading illiquid stock shares. We require that trading ΔN_1 shares of stock within time interval Δt will cost $\alpha|\Delta N_1| = \alpha|u_1|\Delta t$ dollars, where the cost percentage α increases with the rate of trading as $|u_1|^\varepsilon$, with ε being a positive constant. Therefore, trading ΔN_1 shares requires covering a cost equal to $\alpha|u_1|^{1+\varepsilon}\Delta t$ dollars. For simplicity of exposition, we assume that coefficient α is the same for the purchasing and selling of stock 1 shares. The convexity of the transaction costs captures the difficulty of trading shares at a high rate. This convexity does not allow the rate of trading u_1 to be infinite, even though it can be large: the higher the u_1 at fixed ΔN_1, the more costly it is for an investor to trade. Naturally, share holding N_1 becomes absolutely continuous and can be given by expression (4).

The chosen representation of the transaction costs is intuitive. When the market is illiquid, it becomes more difficult for a market maker to find stock shares. The faster shares have to be bought (sold) the more difficult it is to find their seller (buyer). To compensate for facing extra risk and effort, a market maker imposes higher costs on an investor. Consequently, the transaction costs become convex in the rate of trading. Moreover, because u_1 is always finite, an investor is not able to reach the position in stock 1 that maximizes his utility. This situation is observed for many investors in illiquid market. See Isaenko (2005) for a more detailed discussion on the nature of convex transaction costs.

Two remarks are in order. Notice that an investor could have placed a buy (sell) order between many market makers and considerably reduced the nonlinear component in the transaction costs. We assume that the order diversification is limited either by considerable time and effort required to place the orders between many market makers, and/or fixed transaction fees charged by each trader. Second, we assume that transaction charges are placed on the number of shares instead of on the dollar value of the stock traded by an investor. The corresponding modification of our model is straightforward and does not bring any qualitative changes to our conclusions.

2.2 The Investor's Problem

An investor is a price–taker who has CRRA–preferences that support only consumption at time T. The optimization problem faced by this investor is[4]

$$\max_{u_1,\pi_2 \in R^2} \quad E_0\left(\frac{X_T^{1-\gamma}}{1-\gamma}\right) \tag{5}$$

$$\begin{aligned} dX_t &= \left[rX_t + N_{1t}S_{1t}(\mu_1 - r) + X_t\pi_{2t}(\mu_2 - r) - \alpha|u_{1t}|^{1+\varepsilon}\right]dt \\ &+ N_{1t}S_{1t}\sigma_1 dW_{1t} + X_t\pi_{2t}\sigma_2 dW_{2t}, \end{aligned} \tag{6}$$

where $\gamma > 0$, $\gamma \neq 1$, X stands for an investor's wealth and π_2 is the proportion of an investor's wealth allocated to stock 2. Since dX depends explicitly on S_1 and N_1, the description of the problem is completed by adding equations (2) and (4).

Optimization problem (5) can be solved only by the dynamic programming approach. Furthermore, equations (2), (4), and (6) show that the markovian set of state variables in the

[4]We assume that an investor chooses not to sell the stock shares at time T. It is likely that an investor will continue to trade his portfolio after time T but he expects that the market will become liquid and so a new optimization problem will be formulated. Therefore, T could be interpreted as the time during which many stocks will remain illiquid. See also Longstaff (2001).

given economy is (S_1, X, N_1). As in Merton (1969), we define an investor's indirect utility as

$$V(t, S_1, X, N_1) = \max_{u_1, \pi_2 \in R^2} E_t\left(\frac{X_T^{1-\gamma}}{1-\gamma}\right) \tag{7}$$

and find it by solving the corresponding HJB equation.

Notice that we assume that there is no explicit utility maximization over different initial conditions, that is, an investor cannot choose to be at a given point of the state variable space at time zero. This assumption is intuitive: if an investor can trade only at a finite rate and stock returns have infinite first–order variation, the probability that he would have a given allocation is zero and therefore all of them have to be considered.

2.3 Portfolio Choice when the Whole Stock Market is Illiquid

For the purpose of comparison, we present the solution to the portfolio choice problem of an investor in the above market when only a bond and an illiquid stock are available. We assume that the stock price follows $dS_t = S_t(\mu dt + \sigma dw_t)$, where μ, $\sigma > 0$ are constants and w_t is a standard Brownian motion.

If the available stock is liquid, then the portfolio rule and the expected utility are given by

$$\hat{\pi} = \frac{\mu - r}{\gamma\sigma^2}. \tag{8}$$

$$\bar{U} = \frac{X^{1-\gamma}}{1-\gamma} \exp\left\{(1-\gamma)\left[r + \frac{(\mu-r)^2}{2\sigma^2\gamma}\right]T\right\}. \tag{9}$$

$\bar{U}$ is the maximal level of utility which can be achieved in the economy with one illiquid stock. In an economy with two stocks where stock 1 is illiquid, $\bar{U}$ is the minimal possible level of utility given that it is calculated for liquid stock 2. In such a setting, we refer to this level as U_{min}.

Table 1 presents the optimal rate of trading and the expected utility for an investor if the stock market is illiquid. Additionally, these tables show a liquidity premium Δ_l which is the return of the liquid stock that an investor is willing to give up to avoid facing illiquidity:

$$\Delta_l = \mu - r - \sqrt{2\sigma^2\gamma\left(\frac{1}{(1-\gamma)(T-t)} \ln \frac{V}{X^{1-\gamma}/(1-\gamma)} - r\right)}, \tag{10}$$

where V is a value function of an investor in a given market. Detailed discussion of these results can be found in Isaenko (2005).

2.4 Optimal Policies in Liquid Market

For a second benchmark, we describe the optimal strategy of an investor with CRRA–preferences in the market of section 2.1, but in the absence of transaction charges on stock

1. In this case, we refer to the market as being (perfectly) liquid. We find the portfolio rule and the expected utility of an investor

$$\hat{\pi}_i \equiv \frac{1}{\sigma_i\gamma(1-\rho^2)}\left[\frac{(\mu_i - r)}{\sigma_i} - \rho\frac{(\mu_j - r)}{\sigma_j}\right], \qquad i,j = 1,2, \tag{11}$$

$$\begin{aligned} U_{max} &= \frac{X^{1-\gamma}}{1-\gamma}\exp\Bigg\{(1-\gamma)\Bigg[r + \frac{1}{2\gamma(1-\rho^2)} \\ &\times \left(\frac{(\mu_1 - r)^2}{\sigma_1^2} + \frac{(\mu_2 - r)^2}{\sigma_2^2} - 2\frac{(\mu_1 - r)(\mu_2 - r)}{\sigma_1\sigma_2}\rho\right)\Bigg]T\Bigg\}. \end{aligned} \tag{12}$$

Formula (11) implies that holdings of one stock are independent from the presence of the other when the two stocks are independent. As the number of independent stocks increases, the aggregate position in equity of an investor increases proportionally. Nonetheless, the probability for portfolio wealth to fall below a certain threshold diminishes because of diversification. If stocks provide positive risk premia and become more positively/negatively correlated, then an investor takes shorter/longer positions in stocks to reduce his exposure to the correlated parts of risks. If stocks are perfectly correlated and their conditional Sharpe ratios are different, an investor takes infinite opposite positions in two stocks to take advantage of unlimited arbitrage. Therefore U_{max} becomes infinitely large. If $\rho = 1$ and the conditional Sharpe ratios of the two stocks are the same, then an investor is indifferent between the two stocks. Finally, we notice that an investor always chooses a market portfolio since π_i is independent from an investor's wealth.

3 Optimal Policies in Illiquid Market

We now consider the case where transaction costs are present. Furthermore, for clarity of presentation we assume from now on that $\varepsilon = 1$. Appendix A analyzes the problem for arbitrary $\varepsilon > 0$.

First, we identify the conditions for the absence of arbitrage in this economy. To do so, we introduce its definition:

Definition 1 *An arbitrage (a free lunch) is a self–financed portfolio process such that the associated wealth process $X(\cdot)$ satisfies*

$$X_0 \geq 0, \qquad X_T > X_0 e^{rT}. \tag{13}$$

An arbitrage is a portfolio process whose wealth grows locally risklessly with a rate higher than the instantaneous interest rate. Appendix B shows that, similar to perfect markets, arbitrage opportunities are possible only if $\rho = 1$ and $\Delta \equiv \frac{\mu_1 - r}{\sigma_1} - \frac{\mu_2 - r}{\sigma_2} \neq 0$ or if $\rho = -1$ and $\frac{\mu_1 - r}{\sigma_1} \neq -\frac{\mu_2 - r}{\sigma_2}$. We will show that the presence of arbitrage opportunities in this economy does not cause the utility of an investor to become infinitely large and so may sustain for a very long time.

Let us also introduce the liquidity premium Δ_{l1} of stock 1 to be the amount of this stock's conditional return that an investor is willing to give up to avoid trading the illiquid stock. As follows from equation (12), indirect utility is not always a monotonically increasing function of μ_1 when $\rho \neq 0$. Thus, Δ_{l1} is not an accurate description of an investor's welfare when $\rho \neq 0$ so we use it only when $\rho = 0$:

$$V(t, S_1, X, N_1) = \frac{X^{1-\gamma}}{1-\gamma} \exp\left\{ (1-\gamma) \left[r + \frac{1}{2\gamma} \left(\frac{(\mu_1 - \Delta_{l1} - r)^2}{\sigma_1^2} + \frac{(\mu_2 - r)^2}{\sigma_2^2} \right) \right] (T-t) \right\},$$

or

$$\Delta_{l1}(t, S_1, X, N_1) = \mu_1 - r - \sigma_1 \sqrt{2\gamma \left(\frac{1}{(1-\gamma)(T-t)} \ln \frac{V(t, S_1, X, N_1)}{X^{1-\gamma}/(1-\gamma)} - r \right) - \left(\frac{(\mu_2 - r)}{\sigma_2} \right)^2}. \tag{14}$$

The liquidity premium provides an estimate of the additional contribution over the risk premium present in the illiquid stock returns in equilibrium.

One can show that if $|\rho| \neq 1$, then the investor's problem has a solution only if $X \geq N_1 S_1 \geq 0$. The proof of the last result is very similar to that in Longstaff (2001) and is not presented. In essence, an investor cannot borrow, because the illiquid stock can quickly fall and he would not be able to sell enough of this stock to have positive terminal wealth. Furthermore, he cannot short–sell because the illiquid stock can rapidly rise and an investor would not be able to unwind his obligation before his wealth becomes negative. An investor may borrow or take a short position in the illiquid stock if he has an additional source of earnings, for example from labor income, that can hedge the variation in the illiquid stock. Since such earnings are absent in our model and the problem is not defined outside the intervals $X \geq N_1 S_1 \geq 0$, we consider only an economy where $0 \leq \hat{\pi}_1 \leq 1$ unless $|\rho| = 1$. If $|\rho| = 1$, then unpredictable changes in the illiquid stock holdings can be completely hedged away by trading the liquid stock. Thus, an investor can take a short or very long position in the illiquid stock.

We present the HJB equation for the value function $V(T, S_1, X, N_1)$ in Appendix A. First–order conditions in this equation provide optimal policies for an investor:

$$u_1 = \frac{V_{N_1}}{2\alpha V_X} \tag{15}$$

$$\pi_2 = -\frac{V_X(\mu_2 - r) + (V_{XX} N_1 S_1 + V_{XS_1} S_1)\rho\sigma_1\sigma_2}{X V_{XX} \sigma_2^2}. \tag{16}$$

Expression (15) shows that an investor changes the direction of stock 1 trading when $V_{N_1} = 0$. Clearly, an investor can maintain any proportion of π_2 at any time by trading the bond and stock 2 at a possibly infinitely large rate.

Unfortunately, the equation for the indirect utility function cannot be solved analytically and we resort to the traditional finite–difference numerical approach. The following section presents the results of the numerical analysis. We are interested in states of the economy where stock liquidity is limited. Assuming that these states exist for a reasonably short period of time, we set the time horizon for an investor to be only one year and consider the horizon of five years as an exception. In addition, we choose the following parameters unless otherwise specified: $\gamma = 2$, $\sigma_1 = \sigma_2 = 0.2$, $\mu_1 = \mu_2 = 0.07$, and $r = 0.01$.

In the following subsections we consider three cases: when two stocks are independent ($\rho = 0$), when they are imperfectly correlated ($0 < |\rho| < 1$), and when trading stocks provides arbitrage ($\rho = 1$ and $\Delta \neq 0$). One can verify from the following results that the expected utility function of an investor is maximal when $\frac{N_1 S_1}{X} \cong \hat{\pi}_1$ if $0 \leq |\rho| < 1$. This result holds for any positive choices of α and ε. We conclude that an investor barely tries to get rid of the illiquid stock in the presence of the liquid one. Thus, flight to liquidity is negligibly small. Moreover, the maximal expected utility of an investor is very close to that found in the liquid market. These results confirm the crucial importance of analysis of the allocations away from those providing the highest expected utility since the liquidity premium at these allocations could be higher by orders of magnitude than that at allocations with the highest expected utility.

The result that the maximal expected utility of an investor is very close to that in the liquid market is interesting: as soon as an investor reaches the allocations providing this utility he can stop trading and his conditional *expected* utility will be very close to the maximal possible level. However, on rare occasions his allocations to the illiquid stock can fluctuate significantly away from those proving the maximal indirect utility. In the letter case, the role of active trading considerably increases.

Finally, it is straightforward to verify in all of the three cases for the values of ρ, that an investor chooses not to hold the market portfolio, since the weight of each stock holding changes differently with the variations in state variables.

We show results when some of the state variables are fixed. Outcomes for other values of these state variables do not bring any new intuition and are thus not presented.[5] Moreover, we consider the cases when $\alpha = 0.1$ and $\alpha = 0.002$ representing the market with a very strong and moderate illiquidity of stock 1.

3.1 Trading when $\rho = 0$

In the case of uncorrelated stocks, expression (16) becomes $\pi_2 = -\frac{V_X(\mu_2 - r)}{X V_{XX} \sigma_2^2}$. The last result is formally the same as the one in the absence of the illiquid stock. However, given that the

[5]In the calculations, we choose state variable $\ln(S_1)$ instead of S_1 and consider time and state variables $(t, \ln(S_1), X, N_1)$ on the set of values $[0,1] \times [-3,3] \times [0,10] \times [0,3]$, where the corresponding numbers of the grid–points are $10 \times 100 \times 100 \times 800$. The condition $0 \leq N_1 S_1 \leq X$ is maintained on each point of the grid.

indirect utility depends on the state variable N_1, we expect that optimal allocations to the liquid stock would depend on allocations to the illiquid stock.

The last expectation is confirmed in Table 2. Contrary to the case with two liquid stocks, the presence of another stock has an effect on the other's stock trading. In particular, comparison between Table 1 and Table 2 shows that the rate of trading of the illiquid stock increases in the presence of the liquid stock. Managing stock 2 provides additional positive returns and an investor is better off by forwarding some of them towards covering the cost of trading the illiquid stock. As a result, an investor prefers paying higher fees to achieve better diversification of stocks by trading stock 1. On the other hand, the presence of the illiquid stock has an effect on the holdings of the liquid stock: when holdings of the illiquid stock are very low, the proportion of wealth invested in the liquid stock is greater than that when both stocks are liquid, and when holdings of the illiquid stock are too high, the proportion of wealth invested in the liquid stock is lower than that when both stocks are liquid. To explain these allocations, we notice that an investor who maximizes his expected utility function from terminal consumption, tries to increase the probability of high terminal consumption and decrease the probability of low terminal consumption. In perfect markets, this goal is achieved by means of diversification. If one stock is illiquid, then diversification is very restricted. As a result, an investor manages the aggregate risk exposure by directly adjusting his position in the liquid stock. In particular, when aggregate risk exposure is low, an investor increases his allocations to the liquid stock to increase terminal consumption. If aggregate risk exposure is too high, an investor decreases his allocations to the liquid stock to lower the risk of low consumption. The given deviations of π_2 from $\hat{\pi}_2$ diminishes diversification of the stocks. Finally, an investor is much less willing to increase the return and volatility of his portfolio when they are low, than to give up extra volatility at the expense of portfolio returns when the former is too high. This effect follows from the risk aversion of an investor and becomes stronger as this aversion rises. It strengthens even further if stock 1 becomes more illiquid (α increases) or if the volatility of its return rises.

It is interesting that at low values of N_1 the allocations to the liquid stock are not monotonic. Because in the absence of the illiquid stock $\hat{\pi} = 0.75$, they increase from $\pi_2 = \hat{\pi}$ at $N_1 = 0$ and then decrease as $N_1 S_1 / X$ approaches $\hat{\pi}_1$. It turns out that an investor assumes that the illiquid stock is absent for trading when N_1 is very small and chooses π_2 as if only one liquid stock is being traded. If N_1 increases, then an investor considers his position in the illiquid stock as insufficient which he compensates by increasing allocations to the liquid stock.

Table 2 shows that the liquidity premium Δ_{1l} is very small at $N_1 S_1 / X \cong \hat{\pi}_1$ and then increases as $N_1 S_1 / X$ moves away from $\hat{\pi}_1$. Because Δ_{1l} is a discount for the illiquid stock in the presence of the liquid one, it is close to Δ_l which is a similar discount in the economy with only illiquid stock 1 (see Table 1). Still, $\Delta_{1l} < \Delta_l$. The last relation results from the interdependence between the optimal policies of trading two stocks when one stock is illiquid. As an investor gets access to the liquid stock, he improves the management of the risk in illiquid stock holdings which allows him to demand a smaller discount for holding the illiquid stock.

Finally, Table 3 presents the optimal trading, the expected utility function, and the liquidity premium for a longer time horizon ($T = 5$) and two levels of volatility of the illiquid stock returns ($\sigma_1 = 0.2$ and $\sigma_1 = 0.3$). Firstly, we point out a dramatic rise in the trading volume of the illiquid stock for higher T. It is caused by the expected additional returns from both stock holdings which are appreciated by paying higher transaction costs. Secondly, we observe a noticeable change in the allocation π_2 for a longer time horizon: it further increases for low N_1S_1/X and further decreases when this ratio is high. As the terminal consumption becomes more remote, an investor is better off by further adjusting the aggregate risk exposure with stock 2 and lowering the diversification of his position. This is compensated by the increased rate of trading of the illiquid stock allowing an improvement in stock diversification by reaching the neighborhood of $\hat{\pi}_1$ before an investor's horizon. Thirdly, as the time horizon T is extended, an investor can reach $\hat{\pi}_1$ within a smaller time–fraction of his horizon. Hence, the effect of stock illiquidity on the expected utility function (or the liquidity premium) weakens. Table 3 also shows the impact on π_2 coming from a combination of the long–time horizon and the high volatility of the illiquid stock. As seen, π_2 can fall as low as 76% of $\hat{\pi}_2$ at high ratios of N_1S_1/X. Finally, even though stock 2 is very illiquid for a very long time, an investor still maximizes his expected utility at allocations that are very close to those in a liquid market and encounters very small losses in his welfare. Because the rate of trading of the illiquid stock is close to zero in the neighborhood of utility–maximizing allocations, the present value of the transaction costs is small and so are the welfare losses.

3.2 Trading when $0 < |\rho| < 1$

The situation when stocks are completely uncorrelated is rather unusual, even though instructive. In this subsection, we consider the case when $0 < |\rho| < 1$.

Table 4 shows the optimal trading and the expected utility function when $\rho = 0.5$. As follows from equation (11), $\hat{\pi}_1 = \hat{\pi}_2 = 0.5$ so that the expected utility function is highest when $N_1S_1/X \cong 0.5$. Given the results for $\rho = 0$, one would expect π_2 to be reasonably close to $\hat{\pi}_2$. However, we find that π_2 undergoes very significant variations around $\hat{\pi}_2$: if N_1S_1/X is very small, then π_2 is close to $\hat{\pi} = 0.75$, where $\hat{\pi}$ is an optimal proportion of wealth allocated to stock when only a liquid stock is traded, if N_1S_1/X increases but smaller than $\hat{\pi}_1$, then $\pi_2 > \hat{\pi}_2$, and if N_1S_1/X exceeds $\hat{\pi}_1$, then π_2 falls considerably below $\hat{\pi}_2$. The last two patterns of behavior are mainly defined by the correlation between the two stocks. If N_1S_1/X is substantially below $\hat{\pi}_1$, an investor can rely only on stock 2 for risk exposure and will trade as if only this stock is available, choosing $\pi_2 \approx \hat{\pi}$. If N_1S_1/X increases but smaller than $\hat{\pi}_1$, then an investor considers his risk exposure as insufficient and sets π_2 above $\hat{\pi}_2$. The difference between π_2 and $\hat{\pi}_2$ is more significant than in the absence of the correlation between the stocks, since a shorter position in stock 1 substantially decreases the exposure to the correlated risk that can be compensated only by a substantial increase in π_2. If $N_1 S_1/X$ increases above $\hat{\pi}_1$, then the volatility of the portfolio rises and the only way it can be lowered is by selling stock 2 below $\hat{\pi}_2$. Moreover, the leading contributor to the risk exposure becomes the correlated risk. The reduction of this risk requires an investor to sell more liquid stock than he does when stocks are uncorrelated. The given deviations of π_2 from $\hat{\pi}_2$ decrease diversification of the stocks.

The last discussion suggests what happens to π_2 when the correlation between the stocks is negative. Here, the relation between π_2 and $\hat{\pi}_2$ is affected by the size of ρ. Suppose that $N_1 S_1/X > \hat{\pi}_1$. To compensate for the excessive uncorrelated risk of his position in stock 1, an investor decreases his holdings of stock 2. However, to reduce correlated risk, he increases his holdings of stock 2 since it has the dynamics that is opposite to that of the illiquid stock. Therefore, if the correlation is significant, an investor will increase his holdings of stock 2 at an excessive volatility of stock 1 holdings. Similarly, if $0 < N_1 S_1/X < \hat{\pi}_1$, then with a smaller N_1 an investor would like to increase his risk exposure by means of stock 2. If the correlation between the two stocks is small, then the uncorrelated stock plays a leading role causing an increase in π_2. If the correlation is high, then correlated risk plays a leading role making π_2 decrease. Notice that the boundary values of ρ defining different patterns of behavior for the two intervals of $N_1 S_1/X$ are usually different.

So far, we have not made any assumptions about the distribution of the initial allocations of an investor. It seems reasonable that allocations to the illiquid stock at time zero are likely to be above $\hat{\pi}_1$. Often, decreased liquidity of a stock is accompanied by an increased volatility of its returns.[6] A higher volatility of the illiquid stock returns implies smaller allocations to this stock, but as soon as the stock becomes illiquid an investor cannot move to these allocations fast enough and start at proportion of $N_1 S_1/X$ higher than $\hat{\pi}_1$. As a result, he reduces his position in the liquid stock if the correlation between the two stocks is nonnegative and increases this position if the correlation is substantially negative. The former strategy leads to lower prices and higher returns of the liquid stock. This conclusion contradicts a common empirical observation that in the presence of illiquid risky securities investors substantially increase their holdings of liquid risky securities. Because it is likely that the correlation between the two stocks is positive, we explain the discrepancy by the overreaction of investors to the bad news that a lot of stocks have become less liquid. A detailed understanding of the behavior of stock prices requires a general equilibrium analysis of the economy.

Remarkably, the allocations to the liquid stock in Table 4 are essentially the same for both shown values of α even though the trading of stock 1 is negligible if $\alpha = 0.1$ and an investor can reach the neighborhood of $\hat{\pi}_1$ before the end of his time horizon if $\alpha = 0.002$.

[6] According to the Committee on the Global Financial System, during the 1998 crisis, the implied volatility of the S&P500 Index rose from 23% to 43% and that of the thirty–year US T–bond rate from 7% to 14%.

This effect is present if the correlation between the two stocks is significant. To understand its nature, let us decompose two correlated sources of uncertainty to two independent ones [e.g., see Shreve (2004)]: $W_1 = \sqrt{1-\rho^2}B_1 + \rho B_2$, $W_2 = B_2$, where B_1 and B_2 are two independent standard Brownian motions. Then, the wealth of an investor's portfolio follows

$$\begin{aligned} dX_t &= \left[rX_t + \Phi_{\rho t}X_t(\mu_2 - r) + N_{1t}S_{1t}\sigma_1\left(\frac{\mu_1 - r}{\sigma_1} - \rho\frac{\mu_2 - r}{\sigma_2}\right) - \alpha u_{1t}^2 \right] dt \\ &+ N_{1t}S_{1t}\sigma_1\sqrt{1-\rho^2}dB_{1t} + X_t\Phi_{\rho t}\sigma_2 dB_{2t}, \end{aligned} \tag{17}$$

where $\Phi_{\rho t} = \rho\frac{\sigma_1}{\sigma_2}\frac{N_{1t}S_{1t}}{X_t} + \pi_{2t}$.

If both stocks are liquid, then it is easy to show that $\Phi_\rho = \frac{\mu_2 - r}{\gamma\sigma_2^2}$, which is a position in stock 2 when stock 1 is absent. Table 4 shows that the risk exposure Φ_ρ is almost the same for all values of N_1 and very close to $\frac{\mu_2 - r}{\gamma\sigma_2^2}$. *We conclude that for any position in illiquid stock 1, an investor takes a risk exposure in liquid stock 2, such that an aggregate exposure to source of uncertainty B_2 is equal to that in the absence of the illiquid stock.* The exposure to the other source of uncertainty B_1 is defined by the current position in the illiquid stock.

Table 4 shows that the described strategy persists for both values of α. Because Φ_ρ is independent from u_1, π_2 does not change with the illiquidity of stock 1 (parameter α). Thus, π_2 will not change even if this illiquidity becomes small. Moreover, for a given state (S_1, X, N_1), one can easily estimate π_2 in the illiquid market from $\hat{\pi} - \rho\frac{\sigma_1}{\sigma_2}\frac{N_1 S_1}{X}$, where $\hat{\pi}$ is found in the absence of the illiquid stock. We conclude that deviations of allocations to stock 2 from those in the perfectly liquid market could be very considerable even when stock illiquidity is small. Therefore, full consideration of illiquidity is important for finding the best strategies even when this illiquidity may seem to be insignificant. In the latter case, however, an investor reaches the best allocations relatively quickly. As long as the correlation between stocks is substantial, we expect that the given strategy is stable with respect to changes in the parameters of the model, the underlying processes as well as the preferences of an investor.

Knowing a good estimate for π_2 when ρ is substantial, easily allows us to find conditions when the deviation of π_2 from $\hat{\pi}_2$ becomes very strong. For example, for the set of parameters $\mu_1 = \mu_2 = 0.07$, $\sigma_2 = 0.2$, $\gamma = 2$, and $r = 0.01$ this deviation becomes very large at high ρ and/or high σ_1. Table 5 shows the optimal policies and the expected utility when $\rho = 0.8$, $\sigma_1 = 0.2$, and $\rho = 0.5$, $\sigma_1 = 0.3$. If both stocks are liquid, then for the first choice of parameters the optimal allocations are $\hat{\pi}_1 = \hat{\pi}_2 = 0.417$, while for the second choice of parameters they are $\hat{\pi}_1 = 0.111$ and $\hat{\pi}_2 = 0.667$. As seen in Table 5, even at $N_1 S_1/X = 0.5$ the difference between $\hat{\pi}_2$ and π_2 is substantial. It becomes extreme at $N_1 S_1/X \approx 1$ when an investor may take a short position in the liquid stock even though $\hat{\pi}_2 > 0$.

At this point, we notice that our analysis of the case $\alpha = 0.1$ could be related to that of Kahl, Liu, and Longstaff (2003). While some of their conclusions are similar to ours, there are major differences as well. Similar results include the behavior of the optimal allocation π_2 versus illiquid stock return volatility and risk aversion. The differences include the behavior of the expected utility function: in our case, it is maximal when $N_1S_1/X \cong \hat{\pi}_1$, not when $N_1 = 0$ as in Kahl, Liu, and Longstaff (2003). This difference is a consequence of their illiquid stock being a small stock with an idiosyncratic risk. Moreover, contrary to our conclusions, these authors find that the deviation of the optimal portfolio weight π_2 from this weight in the unconstrained case decreases as T increases. The difference is caused by the presence of intertemporal consumption in the model by Kahl, Liu, and Longstaff (2003). In their model, as the stock stays illiquid for a longer time, the intertemporal consumption in the illiquid market becomes more important for an investor. In the meantime, the longer time horizon in our model weakens the effect of the stock illiquidity because an investor has more time to reach the utility–maximizing allocations by the time of terminal consumption. The strengthening of the effects from stock illiquidity in the model by Kahl, Liu, and Longstaff (2003) and their weakening in our model, defines the difference in the behavior of π_2 versus increasing T. Above all, the economic setups in Kahl, Liu, and Longstaff (2003) and our paper are very different: Kahl, Liu, and Longstaff find the portfolio rules of an insider in a perfectly liquid market, while we find the portfolio rules of an investor in an illiquid market.

3.3 Limits to Arbitrage, or Trading when $|\rho| = 1$

Now we analyze the case when $\rho = 1$. The case when $\rho = -1$ is similar and will not be considered. If $\rho = 1$ and the conditional Sharpe ratios of the two stocks are the same, then an investor trades only the liquid stock and the problem becomes trivial. Hence, we assume that $\Delta \neq 0$ and arbitrage is present in the market. The given situation is related to markets where securities providing identical cash flows, but having different liquidities, also have different price dynamics which may allow an arbitrage.

As follows from the proof in Appendix B, the market allows arbitrage in states where N_1 has the same sign as Δ [see equation (B-1)]. This arbitrage is limited by applicable fees. Arbitrage may also be available at states where N_1 and Δ have opposite signs. Notice that the rate of portfolio growth in equation (B-1) of Appendix B cannot be made arbitrarily high since an investor can change N_1 only at a finite rate. Accordingly, the expected utility of an investor is finite even in the presence of arbitrage opportunities. The finiteness of the expected utility causes an investor to consider arbitrage as one of many strategies available for trading, rather than as a dominant one.

We confirm the above expectations by considering an investor's optimal portfolio whose value follows

$$dX_t \quad = \quad \left[rX_t + \Phi_t X_t(\mu_2 - r) + N_{1t}S_{1t}\sigma_1\Delta - \alpha u_{1t}^2 \right] dt + \sigma_2 \Phi_t X_t dW_{1t},$$

where $\Phi_t = \frac{\sigma_1}{\sigma_2}\frac{N_{1t}S_{1t}}{X_t} + \pi_{2t}$ is the cumulative risk exposure of an investor.
Table 6 reports optimal policies at different values of state variable N_1 when stock 1 can have the two levels of illiquidity. We assume that $(\mu_1 - r)/\sigma_1 = 0.1$ and $(\mu_2 - r)/\sigma_2 = 0.3$, so the

illiquid stock is overpriced with respect to the liquid one and $\Delta < 0$. Similar to the above case with $\rho = 0.5$, in all shown states an investor keeps Φ constant at the level corresponding to the market with only liquid stock 2. That is, for any position in the illiquid stock, an investor takes a risk–offsetting position in the liquid stock plus additional positions in this stock and a bond which match those in the absence of the illiquid stock. The first two allocations define a locally riskless strategy whose associated wealth follows $X_a(t) = \int_0^t (\Delta N_1 S_1 \sigma_1 - \alpha u_1^2) d\tau$. The last strategy is an arbitrage for states with a significantly short position in stock 1, since its drift is nonnegative $\forall t \in [0, T]$. It can also be arbitrage for other states N_1 which are not far from $N_1 = 0$. We expect that the described pattern of trading is stable with respect to changes in the parameters of the model, the underlying processes, and the preferences of an investor. Thus, we recover a result similar to that in Basak and Croitoru (2000) who show that in the presence of market frictions an investor considers arbitrage as one of many strategies if taking advantage of arbitrage is limited.[7] Instead of allocating to only arbitrage strategy, an investor also trades a risky strategy. In our case, the frictions are caused by stock illiquidity, while Basak and Croitoru treat liquid markets in the presence of short–sale and long–position constraints. We expect that arbitrage opportunities may exist in a general equilibrium if markets are illiquid.

Notice that an investor offsets the risk of his position in the illiquid stock even when N_1 is significantly positive and an arbitrage strategy is not available. By doing this, he removes the risk related to his position in the illiquid stock which is very difficult to manage by trading the illiquid stock itself.

4 Extension to Multiple Liquid Stocks

In this section we extend our results of Sections 2 and 3 to the market with multiple liquid stocks. That is, we discuss an optimal trading by an investor when liquid and illiquid stocks in the economy can be aggregated into one illiquid and n-1 liquid stocks propelled by n different sources of systematic risk represented by adapted n–dimensional standard Brownian motion $W = (W_1, .., W_n)^T$:

$$\begin{aligned} dS_{1t} &= S_{1t}(\mu_1 dt + \sigma_1 dW_{1t}), \\ & \quad \cdot \; \cdot \; \cdot \\ dS_{nt} &= S_{nt}(\mu_n dt + \sigma_n dW_{nt}), \end{aligned}$$

where all conditional moments of each stock returns are constant, only stock 1 is illiquid, and the covariance matrix of the stocks returns is non–degenerate.

Introduction of multiple liquid stocks allows us to control the size of the market taken by

[7] See also Basak and Croitoru (2006), Liu and Longstaff (2004), and Isaenko (2004).

illiquid stocks. During some market episodes this size is dominant, while most of the time it is not very significant.

The analysis of the portfolio–choice problem in this new economy is a straightforward extension of the analysis in Sections 2 and 3. Because this analysis remains numerical, it is now complicated by the large size of the covariance matrix of stock returns. To preserve a tractability we emphasize the market where the covariance matrix of stock returns is diagonal. In such a market, the rate of trading of the illiquid stock is given by equation (15), while the proportion of wealth invested in each liquid stock is $\pi_i = -\frac{V_X(\mu_i - r)}{XV_{XX}\sigma_i^2}$, $i = 2, ..., n$. These proportions are found after rewriting the HJB equation of Appendix A for multiple stocks and solving it numerically.

Table 7 shows u_1, π_i, $i = 2, ..., n$ and Δ_{l1}, where the last parameter is, as before, the return of stock 1 that an investor is willing to give up to avoid trading the illiquid stock in the given economy. We assume that illiquidity of stock 1 is very strong ($\alpha = 0.1$), $T = 1$, and $n = 5$, 10, and 25. In the limit when n is infinitely large and the diversification role of the illiquid stock is negligible, an investor always tries to sell all his holdings of illiquid stock [see Kahl, Liu, and Longstaff (2003)]. Therefore, as n increases and the illiquid stock becomes less significant for diversification, an investor tries to reduce his allocation to it, so his expected utility is maximal at the proportion of N_1S_1/X that is lower than $\hat{\pi}_1$ and decreases with n. Remarkably, however, the resulting flight to liquidity is small even if n is very large. For example, for $\hat{\pi}_1 = 0.75$ an investor switches the direction of stock 1 trading at $N_1S_1/X = 0.725$, 0.695, and 0.598 at $n = 5$, 10, and 25, respectively. In the last case, illiquid stocks take only 4% of the whole market but an investor tries to reduce his allocation to them only by 20%. If the illiquidity of stock 1 is moderate ($\alpha = 0.002$) and we keep the parametrization of Table 7, then an investor switches the direction of stock 1 trading at $N_1S_1/X = 0.739$, 0.720, and 0.677 at $n = 5$, 10, and 25, respectively.

Table 7 also shows that the presence of illiquid stock shares in an investor's portfolio has a significant effect on his positions in the liquid securities even when n is large. For example, if $\hat{\pi}_i = 0.75$ and $N_1S_1/X = 1.0$ then $\pi_i = 0.714$, 0.711, and 0.704 ($i = 2, ..., n$) at $n = 5$, 10, and 25, respectively. That is, if illiquid stocks take 4% of the whole market, allocation to *each* remaining liquid stock could be substantially different from that in the absence of illiquid stock. The deviation of π_i from $\hat{\pi}_i$ in the given example slightly increases with n because the utility maximizing proportion of N_1S_1/X decreases. Finally, the importance of the illiquid stock for an investor is also seen from the considerable liquidity risk premium that he is willing to give up to avoid trading the illiquid stock. Notice that if proportion of N_1S_1/X is close to 1, then this premium increases with n. However, the maximal possible liquidity premium falls with n since illiquid stock becomes less significant.

The last results are affected by the time–horizon of an investor. In particular, if he expects that a lot of stocks will remain very illiquid for a long time, then the flight to liquidity noticeably increases with n. For example, if $\hat{\pi}_1 = 0.75$, $\alpha = 0.1$, and $T = 5$, then an investor switches the direction of stock 1 trading at $N_1S_1/X = 0.662$, 0.505, and 0.266 at $n = 5$, 10, and 25, respectively. Consequently, the deviation of allocation π_i from $\hat{\pi}_i\,(i = 2, ..., n)$ substantially increases at large and small proportions of N_1S_1/X. The effect from longer illiquidity of stock 1 at fixed n is similar to that from increasing n at fixed T: if an investor's horizon extends and n is large, illiquid stock becomes substantially less significant for risk management and terminal consumption, so the proportion of N_1S_1/X corresponding to the highest expected utility falls. Table 8 illustrates these conclusions for $n = 10$ and $T = 1$, 3, and 5. In particular, it shows that the proportion of N_1S_1/X maximizing the expected utility of an investor falls from $\hat{\pi}_1$ with higher T, while the maximal liquidity premium becomes less substantial.

We conclude that, unless n is very large and many very illiquid stocks will remain such for a very long time (tens of years), presence of these stocks in the market is important for an investor even if these stocks take a very small fraction in the market. It is highly unlikely for this conclusion to depend on the structure of the covariance matrix of stock returns.

It is interesting to extend our approach for analytical estimation of π_2 of Section 3.2 to the similar estimation of π_i, $i = 2, ..., n$ in the multidimensional case. It is known that each standard Brownian motion $W_i\,(i = 1, ..., n)$ can be represented by the linear combination of n independent standard Brownian motions $B_1, ..., B_n$. The latter Brownian motions result from the orthogonal transformation of the original ones. It is also known that $B_1, ..., B_n$ can be chosen such that $B_k = W_k$ for a chosen $k > 1$. It follows from our numerical analysis of subsection 3.2 that, given that the correlations between an illiquid stock and stock k is significant, the risk exposure of an investor with respect to W_k does not change with the allocation to the illiquid stock N_1 and equals to $\frac{\mu_k - r}{\gamma\sigma_k^2}$, while the risk exposure to $B_i\,(i \neq k,\ i > 1)$ generally changes with N_1. Therefore, one can write

$$\frac{\mu_k - r}{\gamma\sigma_k^2} = \rho_{1k}\frac{\sigma_1}{\sigma_k}\frac{N_1S_1}{X} + \sum_{i>1}\rho_{ik}\frac{\sigma_i}{\sigma_k}\pi_i. \tag{18}$$

where the right side is a risk exposure with respect to W_k found by induction.
If there is a liquid stock with a negligible correlation with stock 1 then approximation (18) for this stock deteriorates. We notice however that even if the number of the latter stocks is significant, approximation (18) could be reasonable and used for all liquid stocks. As long as the last claim is correct we can write expression (18) for all $k = 2, ..., n$ to provide us $n - 1$ equations for unknown $\pi_1, ..., \pi_n$. The resulting nonhomogeneous system of equations can be solved by using the Kramer method.

Finally, let us consider the market with arbitrage opportunities. We assume that such opportunities are never available if only liquid securities are traded. Therefore, arbitrage strategy should involve a nontrivial position in the illiquid stock. Moreover, the essence of this strategy remains the same regardless of the number of available liquid stocks: for a given position in the illiquid stock, an investor takes an exactly offsetting position in liquid stocks plus an additional position in the latter stocks providing him a desired risk exposure.

Consequently, we assume that the total number of available stocks is $n+1$ and liquid stock $n+1$ has a perfect correlation with illiquid stock 1, while their conditional Sharpe ratios are different. As above, the covariance matrix of returns of first n stocks is non-degenerate. In this market, u_1 and π_{n+1} are similar to u_1 and π_2 in Section 3.3, while allocations $\pi_2, ..., \pi_n$ are similar to those in the market without arbitrage opportunities. Therefore, one can use the above approach to analytically estimate $\pi_2, ..., \pi_{n+1}$.

The results of this section allow us to conclude that, as long as the fraction of illiquid stocks in the market is noticeable, a lot of intuition about optimal trading of illiquid and liquid stocks can be learned from our analysis of this trading of only two stocks one of which is illiquid.

5 Conclusion

We analyze the optimal behavior of an investor who trades a lot of liquid as well as a lot of illiquid individual stocks. We assume that these stocks can be aggregated into $n-1$ ($n \geq 2$) liquid stocks and one illiquid stock. We find that the role of the illiquid stock is likely to remain significant even if the number of liquid stocks is very large. Therefore, a lot of intuition about portfolio rules in the market with multiple liquid stocks can be learned from the analysis of these rules in the market with only one liquid and one illiquid stock. The rest of conclusions is related to this market.

We find that holdings of both risky securities will be considerably different from those where both stocks are liquid, even if the two stocks are independent. In addition, the trading rate of the illiquid stock increases versus that in the market with only one illiquid stock. If there is a significant correlation between the two stocks and the illiquid stock holdings are small, then an investor always reduces his allocations to the liquid stock to the ones in the market with one liquid stock. If his holdings of the illiquid stock are too high, he adjusts these allocations to decrease cumulative volatility. In particular, if there is a positive correlation between the two stocks, then an investor can take a short position in the liquid stock, even if a perfectly liquid case assumes this position to be significantly positive. If two stocks are independent, then the above effects are present but may be less pronounced. The given adjustments in positions of liquid stock decrease diversification of the stocks. If the correlation between the two stocks is significant (which is likely to be the case), then the position in liquid stock can be estimated from such a position in the market with only a liquid stock.

In the special case where the two traded stocks have perfect correlation, we uncover the story of limits to arbitrage: an investor considers arbitrage opportunities on an equal basis with other investment opportunities and takes a risky position in their presence.

Appendix A

In this appendix we formulate the HJB equation to be solved for the indirect utility function when an investor trades a liquid bond and stock, and one illiquid stock. The analysis is conducted for an arbitrary positive degree of convexity ε.

The value function $V(t, S_1, X, N_1)$ of an investor solves the following PDE

$$\begin{aligned} \max_{u_1,\pi_2\in R^2} \quad & \Bigg\{V_t + \left[\frac{1}{2}(N_1S_1\sigma_1)^2 + \rho N_1S_1\pi_2X\sigma_1\sigma_2 + \frac{1}{2}(\pi_2X\sigma_2)^2\right]V_{XX} + \frac{1}{2}\sigma_1^2S_1^2V_{S_1S_1} \\ + \quad & (N_1S_1\sigma_1 + \rho\pi_2X\sigma_2)\sigma_1S_1V_{XS_1} + \mu_1S_1V_{S_1} + u_1V_{N_1} \\ + \quad & \left[rX + N_1S_1(\mu_1 - r) + \pi_2X(\mu_2 - r) - \alpha|u_1|^{1+\varepsilon}\right]V_X\Bigg\} = 0, \\ & V(T, S_1, X, N_1) = X^{1-\gamma}/(1-\gamma). \end{aligned} \tag{A-1}$$

The first–order condition implies

$$u_1 = \frac{V_{N_1}}{|V_{N_1}|}\left(\frac{|V_{N_1}|}{\alpha(1+\varepsilon)V_X}\right)^{\frac{1}{\varepsilon}}, \tag{A-2}$$

Result (A-2) provides expression (15) if we replace ε with 1. Finding optimal proportion π_2 is straightforward. When optimal u_1 and π_2 are substituted into equation (A-1), we find:

$$\begin{aligned} & V_t + \frac{1}{2}(N_1S_1\sigma_1)^2V_{XX} + \frac{1}{2}\sigma_1^2S_1^2V_{S_1S_1} + N_1S_1^2\sigma_1^2V_{XS_1} + \mu_1S_1V_{S_1} + [rX + N_1S_1(\mu_1 - r)]V_X \\ - \quad & \frac{[V_X(\mu_2 - r) + (V_{XX}N_1S_1 + V_{XS_1}S_1)\rho\sigma_1\sigma_2]^2}{2V_{XX}\sigma_2^2} + \varepsilon\left(\frac{|V_{N_1}|}{\alpha(1+\varepsilon)V_X}\right)^{\frac{1}{\varepsilon}}|V_{N_1}| = 0, \end{aligned} \tag{A-3}$$

where z^+ is a positive part of variable z.

Equation (A-3) can be solved only numerically. Since this equation is highly nonlinear, standard existence and uniqueness results for its solution do not apply. Therefore, we simply assume that the solution exists and that it is unique. Moreover, we assume that our numerical solution converges uniformly to the true solution in the limit of infinitely small space (and so time) increments.

Appendix B

In this appendix we show that arbitrage opportunities in the economy with a bond and two stocks one of which is illiquid are possible only if $\rho = 1$ and $\Delta \neq 0$ or if $\rho = -1$ and $\frac{\mu_1 - r}{\sigma_1} \neq -\frac{\mu_2 - r}{\sigma_2}$.

Indeed, if $\rho = 1$, thenW_1 and W_2 are identical and we can make the volatility of the portfolio to be zero by setting $\pi_2 = -\frac{\sigma_1}{\sigma_2}\frac{N_1S_1}{X}$. Consequently, equation (6) becomes

$$dX_t \quad = \quad \{rX_t + N_{1t}S_{1t}\sigma_1\Delta - \alpha|u_{1t}|^{1+\varepsilon}\}dt. \tag{B-1}$$

If the allocation N_1 has the same sign as Δ, the portfolio return can be made locally riskless and higher than r, because an investor can choose u to be arbitrarily small. Assuming that an investor can be in a state where N_1 and Δ have the same sign, we find access to arbitrage opportunities. If $\Delta = 0$, then the locally riskless rate of portfolio growth is $r - \frac{\alpha|u|^{1+\varepsilon}}{X} \leq r$ and no arbitrage is available. The proof for $\rho = -1$ is similar. Finally, if $-1 < \rho < 1$, then the portfolio with stocks cannot be made locally riskless and arbitrage will not be available.

References

[1] Acharya, V., Pederson, L., 2003, "Asset Pricing with Liquidity Risk." Working paper, New York University.

[2] Amihud, Y., Mendelson, H., 1986, "Asset Pricing and the Bid–Ask Spread." *Journal of Financial Economics*, 17, 223–249.

[3] Amihud, Y., Mendelson, H., 1991, "Liquidity, Maturity, and the Yields on U.S. Treasury Securities." *Journal of Finance*, 46, 1411–1425.

[4] Basak, S., Croitoru, B., 2000, "Equilibrium Mispricing in a Capital Market with Portfolio Constraints." *The Review of Financial Studies*, 13, 715–748.

[5] Basak, S., Croitoru, B., 2006, "On the Role of Arbitrageurs in Rational Markets." *Journal of Financial Economics*, 81, 143–173.

[6] Boudoukh, J., Whitelaw, R. F., 1991, "The Benchmark Effect in the Japanese Government Bond Market." *Journal of Fixed Income*, September, 52–59.

[7] Boudoukh, J., Whitelaw, R. F., 1993, "Liquidity as a Choice Variable: A lesson from the Japanese Government Bond Market." *The Review of Financial Studies*, 6, 265–292.

[8] Brunnermeir, M., Pederson, L., 2005, "Market Liquidity and Funding Liquidity." Working paper, SSRN.

[9] Cetin, U., Jarrow, R., Protter, P., 2004, "Liquidity Risk and Arbitrage Pricing Theory." *Finance and Stochastics*, 8, 311–341.

[10] Cetin, U., Jarrow, R., Protter, P., Warachka, M., 2004, "Pricing Options in an Extended Black Scholes Economy with Illiquidity: Theory and Empirical Evidence." *The Review of Financial Studies*, forthcoming.

[11] Cetin, U., Rogers, L.C.G., 2007, "Modeling Liquidity Effects in Discrete Time."*Mathematical Finance*, 17, 15–29.

[12] Constantinides, G., 1986, "Capital Market Equilibrium with Transaction Costs." *Journal of Political Economy*, 94, 842–862.

[13] Davis, M. H. A., Norman, A.R., 1990, "Portfolio Selection with Transaction Costs." *Mathematics of Operations Research*, 15, 676–713.

[14] Deuskar, P., 2006, "Extrapolative Expectations: Implications for Volatility and Liquidity." Working paper, SSRN.

[15] Duffie, D., Garleanu, N., Pederson, L., 2003a, "Marketmaking in Over–the–Counter Markets." Working Paper, Stanford University.

[16] Duffie, D., Garleanu, N., Pederson, L., 2003b, "Valuation in Over–the–Counter Markets." Working Paper, Stanford University.

[17] Duffie, D., Sun, T. 1990, "Transaction Costs and Portfolio Choice in a Discrete–Continuous Time Setting." *Journal of Economic Dynamics and Control*, 14, 35–51.

[18] Dumas, B., Luciano, E. 1991, "An Exact Solution to a Dynamic Portfolio Choice Problem under Transaction Costs." *Journal of Finance*, 48, 577–595.

[19] Eisfeldt, A., 2004, "Endogenous Liquidity in Asset Markets." *Journal of Finance*, 59, 1–30.

[20] Frey, R., 1998, "Perfect Option Hedging for a Large Trader." *Finance and Stochastics*, 2, 115–141.

[21] Grossman, S., Laroque, G., 1990, "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods." *Econometrica*, 58, 25–52.

[22] Grossman, S., Miller, M., 1988, "Liquidity and Market Structure." *Journal of Finance*, 43, 617–633.

[23] He, H., Mamaysky, G., 2005, "Dynamic Trading Policy with Price Impact." *Journal of Economic Dynamics and Control*, 29, 891–930.

[24] Holmstrom, B., Tirole, J., 2001, "LAMP: A Liquidity–Based Asset Pricing Model." *Journal of Finance*, 56, 1837–1867.

[25] Huang, M., 2003, "Liquidity Shocks and Equilibrium Liquidity Premia." *Journal of Economic Theory*, 109, 104–129.

[26] Huang, J., Wang, J., 2005, "Market Liquidity and Asset Prices under Costly Participation." Working Paper, SSRN.

[27] Isaenko, S., 2004, "Dynamic Equilibrium with Overpriced Put Options." Working Paper, Concordia University.

[28] Isaenko, S., 2005, "Portfolio Choice under Convex Transaction Costs." Working Paper, Concordia University.

[29] Jang, B., Koo, H., K., Liu, H., Loewenstein, M., 2005, "Liquidity Premia and Transaction Costs." Journal of Finance, forthcoming.

[30] Kahl, M., Liu, J., Longstaff, F., 2003, "Paper Millionaires: How Valuable is Stock to a Stockholder who is Restricted from Selling it?" *Journal of Financial Economics*, 67, 385–410.

[31] Kamara, A., 1994, "Liquidity, Taxes, and Short–Term Treasury Yields." *Journal of Financial and Quantitative Analysis*, 29, 403–417.

[32] Lippman, S., McCall, J., 1986, "An Operational Measure of Liquidity." *American Economic Review*, 76, 43–55.

[33] Liu, H., 2004, "Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets." *Journal of Finance*, 59, 289–338.

[34] Liu, H., Loewestein 2002, "Optimal Portfolio Selection with Transaction Costs and Finite Horizons." *The Review of Financial Studies*, 15, 805–835.

[35] Liu, H., Yong, J., 2004, "Option Pricing with an Illiquid Underlying Asset Market." *Journal of Economic Dynamics and Control*, forthcoming.

[36] Liu, J., Longstaff, F., 2004, "Losing Money on Arbitrage: Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities." *The Review of Financial Studies*, 17, 611–641.

[37] Liu, J., Longstaff, F., Pan, J., 2003, "Dynamic Asset Allocation with Event Risk." *Journal of Finance*, 58, 231–259.

[38] Lo, A., X., Mamaysky, H., Wang, J. 2004, "Asset Prices and Trading Volume ander Fixed Transaction Costs." *Journal of Political Economy*, 112, 1054–1090.

[39] Longstaff, F., 1995, "How Much Can Marketability Affect Security Values." *Journal of Finance*, 50, 1767–1774.

[40] Longstaff, F., 2001, "Optimal Portfolio Choice and the Valuation of Illiquid Securities." *The Review of Financial Studies*, 14, 407–431.

[41] Longstaff, F., 2005, "Asset Pricing in Markets with Illiquid Assets." Working Paper, SSRN.

[42] Merton, R. C., 1969, "Lifetime Portfolio Selection under Uncertainty: The Continuous–Time Case." *The Review of Economics and Statistics*, 51, 247–257.

[43] O'Hara, M., 2003, "Presidential Address: Liquidity and Price Discovery." *Journal of Finance*, 58, 1335–1354.

[44] Pereira, P., Zhang, H., 2005, "The Liquidity Premium in a Dynamic Model with Price Impact." Working paper, SSRN.

[45] Rogers, L.C.G., Singh, S., 2006, "Modeling Liquidity and its Effects on Price." Working paper, University of Cambridge.

[46] Shleifer, A., Vishny, R., 1997. The Limits of Arbitrage. *Journal of Finance*, 52, 35–55.

[47] Shreve, S., 2004. *Stochastic Calculus for Finance II: Continuous–Time Models.* New York, Springer–Verlag.

[48] Shreve, S., Soner, H., 1992, "Optimal Investment and Consumption with Transaction Costs" *Annals of Applied Probability*, 4, 609–692.

[49] Vayanos, D., 1998, "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model." *The Review of Financial Studies*, 11, 1–58.

[50] Vayanos, D., 2003, "Flight to Quality, Flight to Liquidity, and the Pricing Risk." Working paper, MIT.

[51] Vayanos, D., Wang, T., 2003, "Search and Endogenous Concentration of Liquidity in Asset Markets." Working paper, MIT.

[52] Weill, O., 2002, "Liquidity Premia in Dynamic Bargaining Markets." Working paper, Stanford University.

Table 1: **Optimal trading of illiquid stock market**

	$\alpha = 0.1$			$\alpha = 0.002$		
NS/X	u	U	Δ_l	u	U	Δ_l
0.020	0.139	-0.9883	0.0431	3.113	-0.9771	0.0141
0.250	0.094	-0.9773	0.0145	2.134	-0.9722	0.0060
0.500	0.047	-0.9702	0.0031	1.072	-0.9690	0.0013
0.748	0.0003	-0.9680	0.0000	0.006	-0.9680	0.0000
0.752	-0.0001	-0.9680	0.0000	-0.003	-0.9680	0.0000
0.850	-0.018	-0.9683	0.0003	-0.413	-0.9681	0.0001
0.925	-0.032	-0.9690	0.0014	-0.731	-0.9684	0.0005
1.000	-0.046	-0.9702	0.0031	-1.044	-0.9689	0.0013

The table reports optimal trading rate u, expected utility function U, and liquidity premium Δ_l when $\gamma = 2$, $\mu = 0.07$, $\sigma = 0.2$, $r = 0.01$, $T = 1$, and $t = 0$, $S = 0.5$, $X = 1.0$, so $\hat{\pi} = 0.75$, and $\bar{U} = -0.9680$.

Table 2: **Optimal trading of two stocks when** $\rho = 0$

	$\alpha = 0.1$				$\alpha = 0.002$			
N_1S_1/X	u_1	π_2	U	Δ_{l1}	u_1	π_2	U	Δ_{l1}
0.020	0.140	0.751	-0.9661	0.0426	3.124	0.752	-0.9552	0.0141
0.250	0.095	0.755	-0.9554	0.0145	2.141	0.752	-0.9504	0.0061
0.500	0.048	0.750	-0.9485	0.0032	1.084	0.749	-0.9473	0.0015
0.748	0.0004	0.737	-0.9465	0.0000	0.007	0.743	-0.9465	0.0000
0.751	-0.0001	0.737	-0.9465	0.0000	-0.010	0.743	-0.9465	0.0000
0.850	-0.019	0.729	-0.9467	0.0006	-0.433	0.740	-0.9466	0.0004
0.925	-0.033	0.723	-0.9475	0.0016	-0.740	0.737	-0.9468	0.0007
1.000	-0.047	0.715	-0.9487	0.0032	-1.063	0.734	-0.9473	0.0015

The table reports optimal policies u_1, π_2, expected utility function U, and the liquidity premium Δ_{l1}, when $\gamma = 2$, $\mu_1 = \mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.5$, $X = 1.0$, so $\hat{\pi}_1 = \hat{\pi}_2 = 0.75$, $U_{max} = -0.9465$, and $U_{min} = -0.9680$.

Table 3: **Optimal trading of two stocks when $\rho = 0$ and $T = 5$**

	$\sigma_1 = 0.2$				$\sigma_1 = 0.3$			
N_1S_1/X	u_1	π_2	U	Δ_{l1}	u_1	π_2	U	Δ_{l1}
0.020	0.363	0.767	-0.8002	0.0161	0.340	0.750	-0.8240	0.0127
0.250	0.261	0.767	-0.7758	0.0059	0.073	0.748	-0.8087	0.0001
0.330	0.219	0.765	-0.7693	0.0035	0.001	0.739	-0.8086	0.0000
0.336	0.218	0.765	-0.7692	0.0035	-0.002	0.739	-0.8086	0.0000
0.500	0.131	0.751	-0.7600	0.0001	-0.132	0.712	-0.8122	0.0021
0.748	0.0002	0.714	-0.7596	0.0000	-0.336	0.648	-0.8327	0.0216
0.752	-0.0005	0.713	-0.7596	0.0000	-0.340	0.647	-0.8333	0.0222
0.850	-0.054	0.692	-0.7599	0.0001	-0.425	0.620	-0.8485	0.0486
0.925	-0.094	0.674	-0.7607	0.0003	-0.491	0.599	-0.8632	NA
1.000	-0.134	0.654	-0.7651	0.0016	-0.550	0.572	-0.8810	NA

The table reports optimal policies u_1, π_2, expected utility function U, and the liquidity premium Δ_{l1}, when $\alpha = 0.1$, $\gamma = 2$, $\mu_1 = \mu_2 = 0.07$, $\sigma_2 = 0.2$, $\rho = 0$, $r = 0.01$, $T = 5$, and $t = 0$, $S_1 = 0.5$, $X = 1.0$, so $\hat{\pi}_2 = 0.75$, $U_{min} = -0.8500$, while $\hat{\pi}_1 = 0.75$, $U_{max} = -0.7596$ for $\sigma_1 = 0.2$, and $\hat{\pi}_1 = 0.333$, $U_{max} = -0.8086$ for $\sigma_1 = 0.3$.

Table 4: **Optimal trading of two stocks when $\rho = 0.5$, $T = 1$ and $\sigma_1 = 0.2$**

	$\alpha = 0.1$				$\alpha = 0.002$			
N_1S_1/X	u_1	π_2	U	Φ_ρ	u_1	π_2	U	Φ_ρ
0.020	0.069	0.740	-0.9671	0.750	1.750	0.740	-0.9639	0.750
0.250	0.035	0.626	-0.9623	0.751	0.924	0.626	-0.9615	0.751
0.498	0.0003	0.500	-0.9608	0.749	0.012	0.500	-0.9608	0.749
0.502	-0.0002	0.498	-0.9608	0.749	-0.001	0.498	-0.9608	0.749
0.750	-0.035	0.370	-0.9623	0.745	-0.901	0.372	-0.9615	0.747
0.850	-0.048	0.318	-0.9640	0.743	-1.274	0.320	-0.9624	0.745
0.925	-0.059	0.279	-0.9657	0.741	-1.543	0.282	-0.9632	0.744
1.000	-0.069	0.240	-0.9677	0.740	-1.813	0.243	-0.9642	0.742

The table reports optimal policies u_1, π_2, and expected utility function U when $\gamma = 2$, $\mu_1 = \mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0.5$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.5$, $X = 1.0$, so $\hat{\pi}_1 = \hat{\pi}_2 = 0.5$, $U_{max} = -0.9608$, and $U_{min} = -0.9680$.

Table 5: **Optimal trading of two stocks.**

	$\rho = 0.8,\ \sigma_1 = 0.2,$			$\rho = 0.5,\ \sigma_1 = 0.3$		
N_1S_1/X	u_1	π_2	U	u_1	π_2	U
0.020	0.027	0.734	-0.9676	0.028	0.735	-0.9676
0.110	0.021	0.661	-0.9667	0.0001	0.668	-0.9672
0.112	0.021	0.659	-0.9667	-0.0006	0.666	-0.9672
0.250	0.011	0.550	-0.9659	-0.042	0.563	-0.9683
0.417	0.0001	0.417	-0.9656	-0.090	0.438	-0.9729
0.419	-0.0001	0.416	-0.9656	-0.091	0.436	-0.9729
0.500	-0.006	0.350	-0.9657	-0.114	0.375	-0.9765
0.750	-0.022	0.151	-0.9669	-0.184	0.187	-0.9927
0.850	-0.029	0.071	-0.9680	-0.212	0.112	-1.0016
0.925	-0.034	0.012	-0.9690	-0.232	0.056	-1.0092
1.000	-0.039	-0.047	-0.9701	-0.254	-0.001	-1.0181

The table reports optimal policies u_1, π_2, and expected utility function U when $\alpha = 0.1, \gamma = 2, \mu_1 = \mu_2 = 0.07, \sigma_2 = 0.2, r = 0.01, T = 1$, and $t = 0, S_1 = 0.5, X = 1.0$. The other parameters are shown at the top of columns 2 and 3. $U_{max} = -0.9656, U_{min} = -0.9680, \hat{\pi}_1 = \hat{\pi}_2 = 0.417$, and $U_{max} = -0.9672, U_{min} = -0.9680, \hat{\pi}_1 = 0.111, \hat{\pi}_2 = 0.667$ in the setting of the second and third columns, respectively.

Table 6: **Optimal trading of two stocks when** $\rho = 1$

	$\alpha = 0.1$			$\alpha = 0.002$		
N_1	u_1	π_2	Φ	u_1	π_2	Φ
-2.500	-0.096	1.992	0.742	-4.892	1.984	0.734
-2.000	-0.096	1.742	0.742	-4.872	1.734	0.734
-1.500	-0.096	1.494	0.744	-4.853	1.488	0.738
-1.000	-0.095	1.246	0.746	-4.834	1.239	0.739
-0.500	-0.095	0.998	0.748	-4.810	0.993	0.741
0.000	-0.094	0.750	0.750	-4.793	0.743	0.743
0.500	-0.094	0.502	0.752	-4.774	0.495	0.745
1.000	-0.094	0.254	0.754	-4.751	0.247	0.747
1.500	-0.093	0.006	0.756	-4.724	-0.003	0.747
2.000	-0.093	-0.242	0.758	-4.701	-0.249	0.749
2.500	-0.092	-0.492	0.758	-4.682	-0.499	0.751

The table reports optimal policies u_1, π_2, and risk exposure Φ when $\gamma = 2$, $\mu_1 = 0.03$, $\mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 1$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.5$, $X = 1.0$.

Table 7: **Optimal trading in illiquid market with multiple liquid stocks**

	$n=5$			$n=10$			$n=25$		
N_1S_1/X	u_1	π_i	Δ_{l1}	u_1	π_i	Δ_{l1}	u_1	π_i	Δ_{l1}
0.020	0.142	0.751	0.0391	0.144	0.752	0.0265	0.154	0.754	0.0178
0.250	0.094	0.755	0.0135	0.093	0.755	0.0115	0.090	0.757	0.0060
0.500	0.044	0.750	0.0027	0.040	0.750	0.0022	0.025	0.750	0.0008
0.597	0.025	0.746	0.0010	0.020	0.745	0.0007	0.0004	0.744	0.0004
0.600	0.025	0.746	0.0009	0.019	0.745	0.0007	-0.0005	0.744	0.0004
0.693	0.006	0.740	0.0002	0.0001	0.739	0.0002	-0.023	0.736	0.0008
0.697	0.006	0.740	0.0002	-0.0001	0.739	0.0002	-0.024	0.736	0.0008
0.723	0.0003	0.738	0.0001	-0.006	0.737	0.0003	-0.031	0.734	0.0011
0.727	-0.0001	0.738	0.0001	-0.007	0.737	0.0003	-0.032	0.734	0.0011
0.850	-0.024	0.728	0.0009	-0.032	0.727	0.0015	-0.061	0.721	0.0033
1.000	-0.053	0.714	0.0043	-0.063	0.711	0.0054	-0.097	0.704	0.0090

The table reports optimal policies u_1, π_i, $i = 2, ..., n$, and liquidity premium Δ_{l1} when $\alpha = 0.1$, $\gamma = 2$, $\mu_1 = \mu_2 = ... = \mu_n = 0.07$, $\sigma_1 = \sigma_2 = ... = \sigma_n = 0.2$, $r = 0.01$, $T = 1$, so $\hat{\pi}_1 = \hat{\pi}_2 = ... = \hat{\pi}_n = 0.75$. The other variables are fixed at $t = 0$, $S_1 = 0.5$, $X = 1.0$ and the covariance matrix of stock returns is diagonal.

Table 8: **Optimal trading in illiquid market with multiple liquid stocks at different T's**

	$T=1$			$T=3$			$T=5$		
N_1S_1/X	u_1	π_i	Δ_{l1}	u_1	π_i	Δ_{l1}	u_1	π_i	Δ_{l1}
0.020	0.144	0.752	0.0265	0.335	0.761	0.0158	0.512	0.764	0.0096
0.250	0.093	0.755	0.0115	0.198	0.763	0.0057	0.304	0.765	0.0038
0.503	0.039	0.750	0.0021	0.056	0.743	0.0010	0.002	0.751	0.0008
0.506	0.038	0.749	0.0021	0.055	0.743	0.0010	-0.001	0.750	0.0008
0.600	0.019	0.744	0.0006	0.001	0.731	0.0006	-0.079	0.729	0.0011
0.603	0.018	0.744	0.0006	-0.0004	0.730	0.0006	-0.081	0.728	0.0011
0.693	0.0001	0.739	0.0002	-0.050	0.715	0.0009	-0.114	0.703	0.0018
0.697	-0.0001	0.739	0.0002	-0.052	0.714	0.0009	-0.115	0.702	0.0019
0.850	-0.032	0.727	0.0015	-0.136	0.681	0.0032	-0.150	0.660	0.0034
1.000	-0.063	0.711	0.0054	-0.219	0.642	0.0078	-0.277	0.621	0.0080

The table reports optimal policies u_1, π_i, $i = 2, ..., 10$, and liquidity premium Δ_{l1} when $n = 10$, $\alpha = 0.1$, $\gamma = 2$, $\mu_1 = \mu_2 = ... = \mu_{10} = 0.07$, $\sigma_1 = \sigma_2 = ... = \sigma_{10} = 0.2$, $r = 0.01$, $T = 1$, so $\hat{\pi}_1 = \hat{\pi}_2 = ... = \hat{\pi}_{10} = 0.75$. The other variables are fixed at $t = 0$, $S_1 = 0.5$, $X = 1.0$ and the covariance matrix of stock returns is diagonal.