Limitations on the Effectiveness of Forward Guidance at the Zero Lower Bound

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Abstract

The recent literature on monetary policy in the presence of a zero lower bound on interest rates has shown that forward guidance regarding the path of interest rates can be very effective in preserving macroeconomic stability in the face of a contractionary demand shock; moreover, that literature apparently leaves little scope for any further improvements in stabilization performance via nontraditional monetary policies. In this paper, we characterize optimal policy under commitment in a prototypical New Keynesian model and examine whether those conclusions are sensitive to the specification of the shock process and to the interest elasticity of aggregate demand. Although forward guidance is effective in offsetting natural rate shocks of moderate size and persistence, we find that the macroeconomic outcomes are much less appealing for larger and more persistent shocks, especially when the interest elasticity parameter is set to values widely used in the literature. Thus, while forward guidance could be sufficient for mitigating the effects of a "Great Moderation"-style shock, a combination of forward guidance *and* other monetary policy measures—such as large-scale asset purchases—might well be called for in responding to a "Great Recession"-style shock.

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1 Introduction

Recent studies of optimal monetary policy at the zero lower bound (ZLB) have focused principally on the benefits of *forward guidance* regarding the anticipated future path of short-term nominal interest rates, with only subsidiary treatment of nontraditional monetary policy measures.[1] This approach suggests that policymakers should announce that the policy rate will be kept low during the initial stages of economic recovery. Such a commitment can provide stimulus to the economy by lowering expected future real interest rates. The expectations of lower real rates arise from two sources in a New Keynesian model: from expectations of low future nominal interest rates, and from higher expected rates of future inflation (Woodford, 1999, p. 302). Eggertsson and Woodford (2003), Nakov (2008), and Walsh (2009) have consequently emphasized the extent to which forward guidance is very effective in stabilizing the output gap and inflation—that is, this policy avoids deflation in the near term while producing only mildly elevated rates of inflation in subsequent periods.

In responding to the recent economic downturn, however, several major central banks have deployed nontraditional measures aimed at providing additional macroeconomic stimulus even though the near-term path of the policy rate may be constrained by the zero lower bound.[2] For example, the Bank of England and the Federal Reserve have engaged in large-scale asset purchases (LSAPs), though with somewhat different emphases which, as Bean (2009, p. 22) notes, partly reflect differences in the financial market structures across these two economies.[3] Nonetheless, the prevailing consensus—as elegantly summarized by Walsh (2009)—casts doubt on the merits of nontraditional policies. Simply put, the key question is: Why not simply rely on forward guidance, since it could deliver all the stimulus required at the ZLB at relatively little cost?

[1] Several early studies focused on the use of simple feedback rules for conducting monetary policy at or near the ZLB; see Wolman (1998), Reifschneider and Williams (1999), and Coenen and Wieland (2003). For analysis of optimal policy under commitment at the ZLB, see Jung, Teranishi, and Watanabe (2001, 2005), Adam and Billi (2003, 2006), Eggertsson and Woodford (2003), and Nakov (2008).

[2] Over the past two years, many central banks have also acted vigorously to fulfill their function as lender-of-last-resort; see Madigan (2009) for further discussion. In this paper, however, we focus on monetary policy actions undertaken with the specific aim of providing macroeconomic stabilization rather than principally aimed at aiding day-to-day financial market functioning.

[3] Because it includes central bank purchases of government assets, our conception of nontraditional monetary policies is broader than that of Gertler and Karadi (2009), who refer to "unconventional monetary policy."

Our analysis in this paper addresses this question by considering optimal policy under commitment in a prototypical New Keynesian model and examining the extent to which the stabilization performance of forward guidance depends on the specification of the shock process and on the interest elasticity of aggregate demand. Throughout this analysis, we abstract from issues that could arise under imperfect credibility and focus on the case—as in nearly all of the existing ZLB literature—where the central bank has a perfect commitment technology.

Although forward guidance is quite effective in offsetting a natural rate shock of moderate size and persistence, we show that macroeconomic outcomes are much less satisfactory for a larger and more persistent shock, especially when the interest elasticity parameter value used in many previous studies is selected. Even though the outcomes under the optimal commitment are preferable to those under discretion, the economy still experiences a steep initial decline in output and a marked swing in the inflation rate. Thus, while forward guidance may be sufficient to mitigate the effects of a "Great Moderation"-style shock, a combination of forward guidance *and* other policy measures—such as nontraditional monetary policies—might well be called for in responding to a "Great Recession"-style shock of the kind economies have recently faced.

Our analysis also points toward reconsidering other aspects of the prevailing consensus regarding monetary policy strategies at the ZLB. First, we find that a policy that targets a constant value for a linear combination of the log price level and the output gap does *not* necessarily provide a close approximation to the optimal commitment policy. This contrasts with the finding by Eggertsson and Woodford (2003) that a constant target for what they call the "output-gap adjusted price index" closely replicates the optimal commitment. When the economy is hit by a large and persistent natural rate shock, the optimal policy involves a commitment to a persistently elevated inflation rate that pushes down the *ex ante* real interest rate and thereby dampens the initial impact of the shock. Our AR(1) specification of the shock implies that the natural rate takes its most negative value in the impact period of the shock. The optimal policy response implies that inflation takes place immediately, and is not merely compensating for a prior period of deflation. In contrast, the constant price level targeting rule generates an initial phase of deflation—with a much steeper drop in output than under the optimal policy—and a subsequent phase of positive inflation that eventually brings the price level back to target.

Second, the optimal policy path does *not* necessarily involve a sharp tightening once the policy rate moves up from the ZLB. As noted by Walsh (2009), this feature is characteristic of the optimal policy path obtained by Eggertsson and Woodford (2003), who focused on two-state Markov shocks, and by Adam and Billi (2003, 2006) and Nakov (2008), who focused on relatively transitory autoregressive (AR) shocks. In contrast, our analysis shows that the pace of policy tightening may be quite gradual if the natural rate shock is large and follows a more persistent AR process.

The remainder of this paper is organized as follows. Section 2 highlights some key features of the recent economic downturn and monetary policy responses in six major industrial economies. Section 3 discusses the basic mechanics of forward guidance at the ZLB. Section 4 reviews the methodology for characterizing optimal policy under commitment in the presence of the ZLB. Section 5 quantifies the limitations of forward guidance. Section 6 considers the stabilization performance of constant price-level targeting. Section 7 presents further sensitivity analysis of these results. Section 8 extends the analysis to the case of uncertainty about the pace of recovery. Section 9 offers some brief concluding remarks.

2 The Recent Experiences of Six Industrial Economies

In this section we outline some of the features of the recent economic downturn and the monetary policy response.

2.1 The Magnitude and Persistence of the Economic Downturn

Table 1 depicts the OECD estimates and forecasts for the output gap in six economies for 2008-2010, as given in the June 2009 issue of its *Economic Outlook*. The table brings out the scale and speed of the deterioration in economic activity. In 2008, no country's output stood more than 0.5 percent below potential.[4] No country is projected to have an output gap *less* negative than minus 4.7 percent in 2009. In no economy is the deterioration in the output gap from 2008 to 2009 less than four percentage points, and in two economies (Japan

[4]These numbers refer to annual averages, so the deterioration late in 2008 is recorded primarily in the 2009 gap estimate.

and Sweden) it is greater than seven percentage points. Moreover, the OECD's projections as of June 2009 suggest that no economy will experience an improvement in the output gap in 2010.

Table 1. Recent and Projected Output Gaps
(*OECD Economic Outlook*, June 2009)

	2008	2009	2010	Std dev (91-07)
United States	-0.5	-4.9	-5.4	1.3
Euro Area	0.4	-5.5	-6.0	1.2
Japan	1.3	-6.1	-6.1	2.0
United Kingdom	0.4	-5.4	-6.4	1.3
Canada	-0.4	-4.7	-5.4	2.1
Sweden	-0.1	-7.7	-8.7	2.1

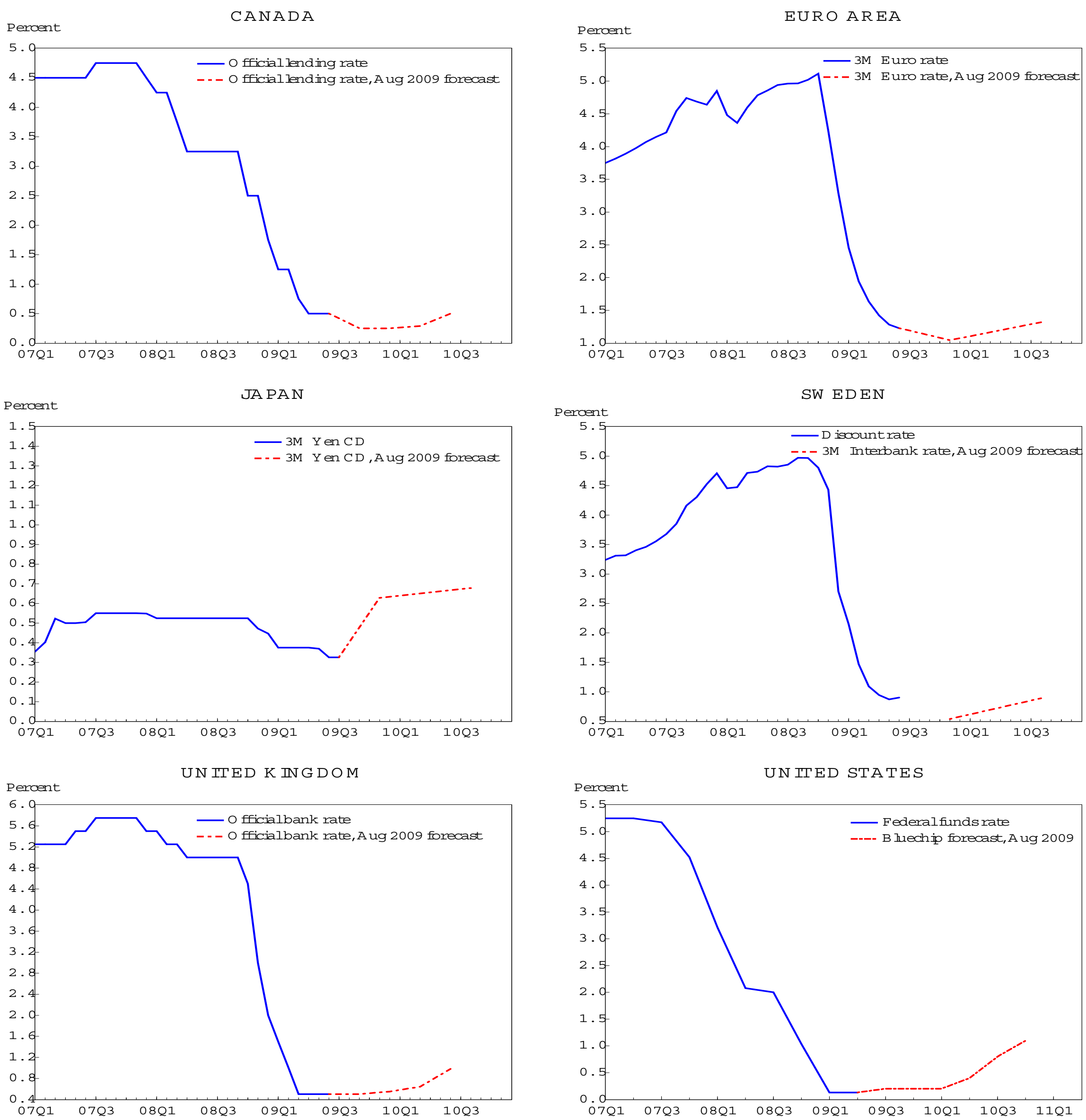
Some longer-term perspective on this change in circumstances is provided by the final column of Table 1, which gives the standard deviation of the output gap for each economy using quarterly OECD gap estimates for the period 1991 to 2007. For each economy, the deterioration in the output gap from 2008 to 2009 is large relative to the historical standard deviation. The shift amounts to two standard deviations for Canada; more than three standard deviations for the United States, Japan, and Sweden; and more than four standard deviations for the euro area and for the United Kingdom. These numbers testify to the scale of the shock that has hit the world economy. The shock, and the associated policy response, are also reflected in the major revisions that occurred between 2007 and 2009 in forecasts of inflation and short-term interest rate in every economy, as shown in Figures 1 and 2.

2.2 Forward Guidance Measures

The reinforcement of current policy actions with signals about the future course of policy is at the heart of the forward guidance approach. As King (1994) observes, the rational expectations revolution highlighted the role of private sector expectations as a major conduit through which monetary policy affects aggregate demand.[5] King (1994) further notes that

[5]The dual insights that long-term interest rate behavior reflects expectations about future policy, and that official signals about future short-term interest-rate policy can contribute to economic stabilization, are a longstanding feature of discussions of monetary policy, predating even the use of forward-looking models

Figure 1: Interest Rate Expectations



Source: US forecast: Blue Chip Financial Forecasts, Vol. 28, No. 8, August 1, 2009; forecasts of federal funds rate. Other countries: Consensus forecasts, August 10, 2009 (Japan, three-month yen CD; euro area 3-month interest rates; U.K., bank rate; Canada, overnight lending rate; Sweden, 3-month interbank rate.)

the U.S. monetary policy tightening sequence of 1994–95—at the onset of which the FOMC started announcing the target federal funds rate—triggered long-term interest rate responses that were "related in important ways to expectations about future policy."

The FOMC statement issued in August 2003 marked the first occasion on which the Federal Reserve gave forward guidance about the likely evolution of its funds rate target. At that point, the FOMC maintained the target federal funds rate at 1 percent and stated that "the risk of inflation being undesirably low is likely to be the predominant concern for the future. In these circumstances, the Committee believes that policy accommodation can be maintained for a considerable period." The Minutes of that FOMC meeting indicate that "[w]hile the Committee could not commit itself to a particular policy course over time, many of the members referred to the likelihood that the Committee would want to keep policy accommodative for a longer period than had been the practice in past periods of accelerating economic activity." Although the description of the inflation outlook varied in subsequent FOMC statements, the "considerable period" language was retained through the end of 2003.

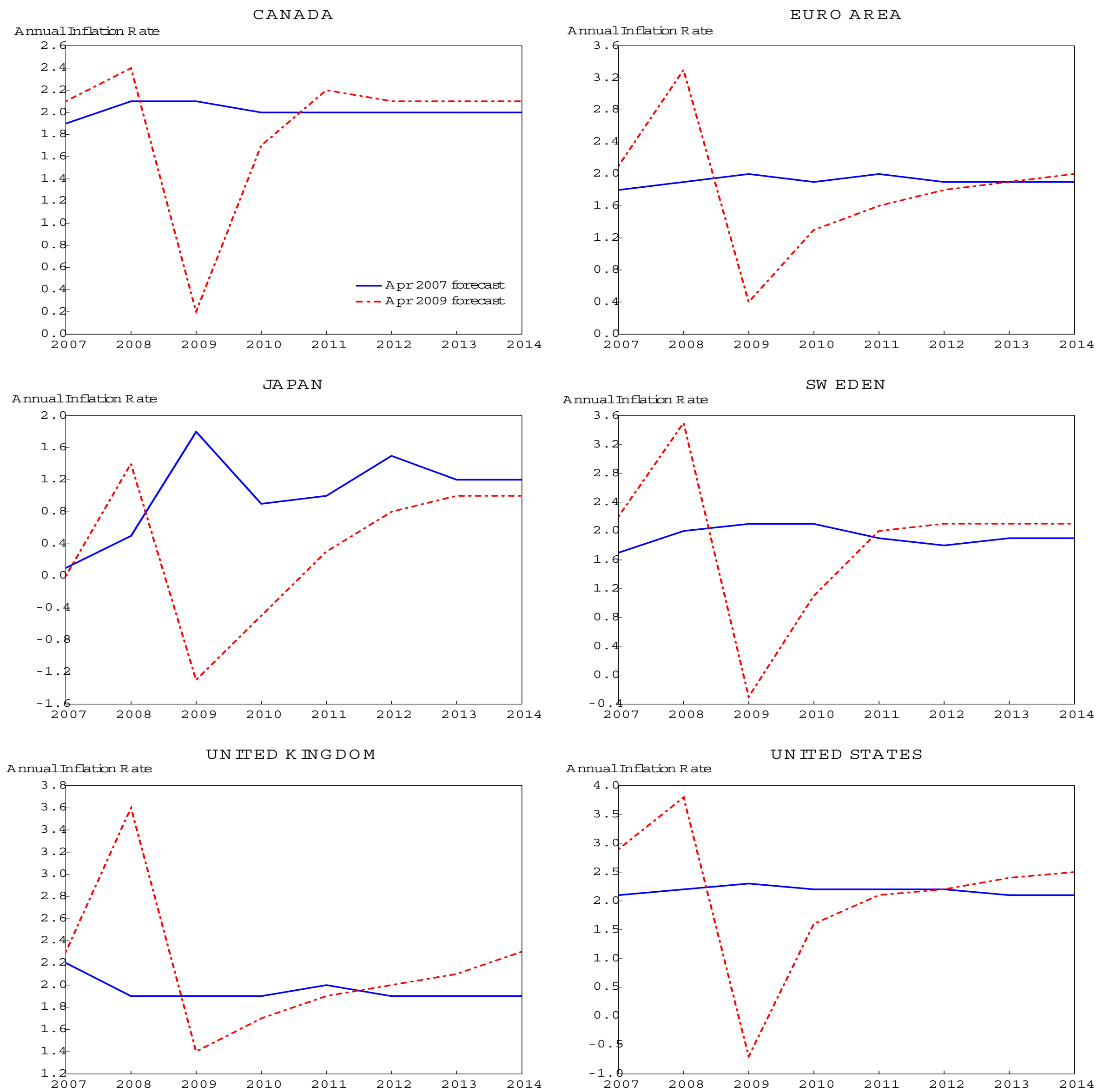
From May 2004 through the end of 2005, FOMC statements indicated that "...the Committee believes that policy accommodation can be removed at a pace that is likely to be measured." The FOMC also underscored the conditional nature of this forward guidance by stating that "...the Committee will respond to changes in economic prospects as needed to fulfill its obligation to maintain price stability." This conditionality was introduced in June 2004—the point at which the FOMC began steadily raising the target federal funds rate by 25 basis points per meeting until this rate reached 5.25 percent in June 2006.[6]

Since December 2008, the FOMC has maintained a target range of 0 to 1/4 percent for the funds rate and has included forward guidance in each of its statements. The December 2008 FOMC statement referred to the likelihood that economic conditions would "warrant exceptionally low levels of the federal funds rate for some time," and in March 2009 this language was adjusted to refer to keeping rates low "for an extended period." Over the past year, the Sveriges Riksbank and the Bank of Canada have also included forward guidance

in macroeconomics; see, for example, Keynes (1930), Simmons (1933), and Radcliffe Committee (1959, para. 447).

[6]From January-April 2004, the FOMC maintained an unchanged funds rate target of 1 percent and stated, "With inflation low and resource use slack, the Committee believes that it can be patient in removing its policy accommodation." The conditionality language was included in each FOMC statement from June 2004 through November 2005.

Figure 2: Inflation Expectations



Source: Consensus Economics, April 2008 and April 2009; long-term forecasts of consumer prices in United States, Japan, euro area, U.K., Canada, and Sweden.

in their policy communications.[7] For example, the July 2009 *Monetary Policy Report* of the Sveriges Riksbank states,"The repo rate will not be raised again until the second half of 2010." (p. 7)

As shown in Figure 1, surveys of professional forecasters indicate that short-term nominal interest rates are expected to follow a fairly shallow path in the United States, Canada, and Sweden. It should be noted, however, that the anticipated path of short-term rates is fairly similar for three other industrial economies—that is, the euro area, Japan, and the United Kingdom—where central bank communications have *not* emphasized forward policy guidance.[8]

Figure 2 depicts professional forecasters' inflation expectations for each of these six industrial economies. Longer-run inflation expectations appear to have remained relatively well-anchored despite the global economic downturn; that is, consumer inflation in each economy is projected to settle over the longer run at rates close to those which were anticipated in spring 2007, prior to the onset of the financial market turmoil. Nonetheless, as noted by Walsh (2009), the medium-term trajectory for inflation does *not* seem to be consistent with the existing literature on optimal policy under commitment, which prescribes an inflation path that rises *above* the long-run goal and remains elevated for an extended period.

2.3 Nontraditional Monetary Policies

We use the term "nontraditional monetary policies" to refer to monetary policy operations in additional assets beyond the "traditional" focus on short-term government securities. As noted in the introduction, a variety of nontraditional policies have been deployed since the onset of the economic downturn.

In late November 2008, the Federal Reserve announced that it would use "all available tools" to promote economic recovery and preserve price stability. At that time and in subsequent FOMC announcements, the Federal Reserve indicated that it would "provide support to mortgage lending and housing markets" by purchasing up to $1.25 trillion in

[7]For further discussion, see Bean (2009).

[8]For more information on interest rate expectations in Japan and their relationship to monetary policy commitment, see Nakajimay, Shiratsukaz, and Teranishi (2009).

agency mortgage-backed securities and up to $200 billion in agency debt. In March 2009, the FOMC announced purchases of up to $300 billion in Treasury securities "to help improve conditions in private credit markets." The Federal Reserve also launched the Term Asset-Backed Securities Loan Facility [TALF] "to facilitate the extension of credit to households and small businesses."

A number of other central banks have also initiated and expanded their nontraditional policy measures over the past year. In December 2008, the Bank of Japan announced that it would accelerate the pace of its purchases of long-term Japanese government bonds (JGBs) and that these purchases would be expanded to include 30-year bonds and inflation-indexed government securities. In early March 2009, the Bank of England announced that it planned a maximum of £75 billion in unsterilized purchases of U.K. long-term government securities; about two months later, the upper limit on those purchases was increased to £125 billion. Finally, the European Central Bank also announced that up to €60 billion purchase of euro-denominated covered bonds, with purchases scheduled for July 2009-June 2010.

The remainder of this paper will consider the limits to forward guidance and the extent to which such limits may serve as an significant rationale for nontraditional policy measures. Although our paper is not aimed at modeling or quantifying the effects of nontraditional policies, a few brief comments may be helpful at this point.

First, while economists have become accustomed to working with models that feature traditional short-term interest rate policy, nontraditional policies are not new from a longer-term perspective; the notion that central bank operations in longer-term debt markets can affect long-term interest rates for a *given* path of expected short-term interest rates has a venerable history, both in central banking (e.g., Riefler, 1958) and the research literature (e.g., Modigliani and Sutch, 1966). The fact that nontraditional policies have no role in the modern consensus model may reflect a gap in the modern research agenda rather than an inherent problem with these policies.

Second, while the empirical evidence in support of nontraditional policies has been questioned—for example, Goodhart (1992, p. 327) states that "studies of the effect of relative debt supplies (at the long, medium and short end) on the yield curve have not found any strong, significant effect," and similar observations have been made by Woodford (1999) and Walsh (2004, 2009)—evidence from the 2000 U.S. Treasury refinancing supports the existence of noticeable effects of nontraditional policies; see Bernanke, Reinhart, and Sack (2004). Moreover, as Sims (1992, p. 975) observes, while many economists view monetary policy's effects solely through the nominal short-term interest rate channel, "the profession as a whole has no clear answer to the question of the size and nature of the effects of monetary policy on aggregate activity." Judgments on the empirical support for nontraditional policies should therefore remain open.

Third, the argument about nontraditional policies' effects is a *ceteris paribus* argument. Central bank purchases of long-term debt might provide downward *pressure* on long-term rates, and thereby contribute to an improvement in the economic outlook; but an improved outlook will in turn tend to raise long-term rates, and is thought to have done so since early 2009.[9] For the United Kingdom, Bean (2009) argues that the *ceteris paribus* effect of the Bank of England's long-term purchases has been substantial, contending that long-term "yields appear to be some 50-75 basis points lower than they would otherwise be." If estimates of effects if this order of magnitude endure, then nontraditional monetary policies would appear capable of making a major contribution to economic stabilization by influencing private credit conditions.

3 The Mechanics of Forward Guidance at the ZLB

In this section, we use the standard New Keynesian model to characterize the basic mechanics of forward guidance when an exogenous decline in aggregate demand leads to the policy rate being pinned at the zero lower bound. We assume that an unanticipated exogenous shock at time $t = 0$ causes the natural real interest rate (r_t^n) to drop below zero and to remain negative until period $t = N$ and then returning to positive values from period $N + 1$ onwards. Once the shock in period zero has shifted the natural rate, the entire new exogenous path of the natural rate is known to agents. The period of negative natural rates is assumed to trigger a sustained zero policy rate, and it is possible that the zero-rate policy may prevail for some periods beyond the duration of negative natural rates; that is, the nominal policy rate may

[9] An early articulation of this point appeared in Friedman and Meiselman (1963, p. 221). They argued that a zero net effect of a monetary injection on market interest rates could be a reflection of the power of monetary policy, since anticipation of higher spending streams from the monetary stimulus could generate upward, offsetting pressure on interest rates.

be set to zero even after the natural rate resumes positive values.

Because of forward-looking private sector behavior, the dynamics of the output gap and inflation over periods 0 to N depend on the expected values of these variables in period $N+1$. The vector $[x_{N+1}, \pi_{N+1}]$—hereafter referred to as the *forward guidance vector*—summarizes the degree to which monetary policy, by managing expectations, helps stabilize the economy over periods 0 to N.

3.1 The Loglinear Model

Our analysis focuses on the same stylized New Keynesian model that has been used in many previous studies. The underlying nonlinear framework, from which this standard loglinearized model is derived, is presented in our appendix. We assume that the economy's steady state is Pareto optimal—that is, the output gap and inflation rate are equal to zero in the absence of shocks. The loglinearized system has only two behavioral equations: a forward-looking New Keynesian Phillips curve (NKPC) and an optimizing IS equation.[10] With Calvo (1983)-style staggered price setting, the NKPC has the following form:

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa x_t \tag{1}$$

where $0 < \beta < 1$ and $\kappa > 0$.

As in Woodford (1999), we express the IS equation in terms of output gaps and interest rate gaps:

$$x_t = E_t\{x_{t+1}\} - \sigma E_t\{i_t - \pi_{t+1} - r_t^n\}, \tag{2}$$

where x_t is the output gap, i_t is the net nominal interest rate, π_t is inflation, r_t^n denotes the natural real rate of interest, and $\sigma > 0$ is the interest elasticity of real aggregate demand, capturing intertemporal substitution in private spending.[11]

[10] See, for example, King (2000) for further discussion.

[11] As we show in the appendix, movements in the natural rate can be related to variations in the underlying real shocks in the model, corresponding to shocks to technology and government purchases.

3.2 Understanding the Mechanics

It is convenient to start from the algebraic representation of the system given in Jung, Teranishi, and Watanabe (2005). For the block of periods over which the policy rate is pegged at zero ($i_t = 0$ for $t = 0, \cdots, N$), equations (1) and (2) can be combined to obtain the following first-order expression for the vector $[\pi_t, x_t]$:

$$\begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = A \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} + \sigma \begin{pmatrix} \kappa \\ 1 \end{pmatrix} r_t^n, \tag{3}$$

where the 2×2 matrix A is defined as

$$A = \begin{pmatrix} \sigma\kappa + \beta & \kappa \\ \sigma & 1 \end{pmatrix}. \tag{4}$$

The two eigenvalues of this matrix, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, satisfy the following quadratic formula:

$$\tilde{\lambda} = \frac{\sigma\kappa + \beta + 1 \pm \sqrt{(\sigma\kappa + \beta + 1)^2 - 4\beta}}{2}. \tag{5}$$

Given that $0 < \beta < 1$, $\kappa > 0$, and $\sigma > 0$, it is straightforward to verify that both eigenvalues of A are real and that one is explosive while the other is stable.

Equation (3) can be iterated forward to yield the outcomes for inflation and the output gap as a function of the expected values of these variables at period $N + 1$:

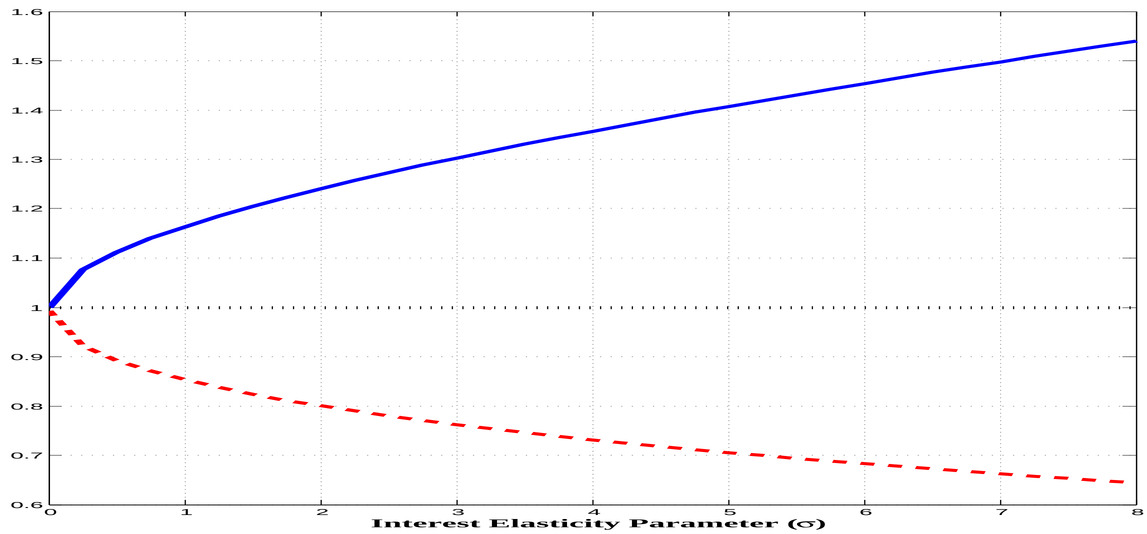
$$\begin{pmatrix} \pi_{N-j} \\ x_{N-j} \end{pmatrix} = A^{j+1} \begin{pmatrix} \pi_{N+1} \\ x_{N+1} \end{pmatrix} + \sigma \sum_{i=0}^{j} A^i \begin{pmatrix} \kappa \\ 1 \end{pmatrix} r_{N-j+i}^n \tag{6}$$

for $j = 0, \cdots, N$.

Evidently, the central bank faces intrinsic limits in seeking to stabilize the economy over the span of time periods for which the policy rate is pinned at the ZLB. In particular, the forward guidance vector $[x_{N+1}, \pi_{N+1}]$ serves as a terminal condition in period $N+1$ that pins down the rational expectations equilibrium for the preceding periods. Nonetheless, looking backwards from period $N + 1$, the economy behaves almost like an "uncontrolled" dynamic system—which is to say, one with a tendency to diverge instead of exhibiting dynamic stability—and depends on the vector $[x_{N+1}, \pi_{N+1}]$, the exogenous path of the natural rate, and the transition matrix A.

Equation (6) highlights several key, interrelated, principles regarding the impact of forward guidance on the dynamics of output and inflation at the ZLB:

Figure 3: The Mechanics of Forward Guidance



Note: This figure depicts how the interest elasticity parameter σ influences the magnitudes of the two eigenvalues that determine the behavior of the uncontrolled system through the period over which the policy rate is held at the ZLB. The solid and dashed lines represent the eigenvalues of matrix A. This figure uses the baseline parameterization of $\beta = 0.9925$ and $\kappa = 0.024$.

First, forward guidance *cannot* deliver a constant degree of aggregate demand stimulus over the block of periods where policy is constrained by the ZLB. In particular, for a given period $N-j$, the magnitude of stimulus implied by the forward guidance vector $[x_{N+1}, \pi_{N+1}]$ depends on the matrix A^{j+1} and hence on the eigenvalues of A. From (5), it is apparent that this pair of eigenvalues only lies on the unit circle if $\beta = 1$ and either $\kappa = 0$ or $\sigma = 0$, in which case the matrix A would be idempotent, that is, $A^j = A$ for all $j > 0$. The parameter combinations required to deliver a constant degree of stimulus is therefore ruled out by the parameter spaces implied by the New Keynesian model.

Second, the evolution of the economy at the ZLB depends crucially on the interest elasticity parameter (σ). Figure 3 depicts the pair of eigenvalues of the "uncontrolled" system as a function of σ, using our benchmark parameterization of $\beta = 0.9925$ and $\kappa = 0.024$. When σ has a relatively small value of 0.5, as in Eggertsson and Woodford (2003), both eigenvalues are reasonably close to unity, so that the matrix A is nearly idempotent. In contrast, when σ has a value of about 6, as in Rotemberg and Woodford (1997) and Woodford (2003), the explosive eigenvalue exceeds 1.4 and hence the "uncontrolled" economy exhibits highly unstable dynamics.[12]

Third, the degree of stability of the macroeconomy is influenced by the value of N, that is, by how long the policy rate stays at the ZLB. In particular, for any given magnitudes of the two eigenvalues of A, the natural rate shock and the forward guidance vector will have larger effects at time 0 on the inflation rate and the output gap, which depend on the matrix A raised to the power $N+1$.

Finally, the effectiveness of forward guidance depends on the evolution of the natural rate over the block of periods that this rate remains below zero. For example, the contractionary impact of the shock at time zero will tend to be heightened if the natural rate follows an AR(1) process, as in Jung, Teranishi, and Watanabe (2001, 2005), Adam and Billi (2003, 2006), and Nakov (2008).

In contrast, forward guidance tends to promote more stable paths of inflation and the output gap in the case where the natural rate shock follows a two-state Markov-switching process, as in Eggertsson and Woodford (2003). Indeed, if A is nearly idempotent and $r_t^n = -\tilde{r}^n$ for periods $t = 0, \cdots, N$, then forward guidance can provide nearly perfect stabilization outcomes (i.e., it can render the ZLB constraint almost nonbinding), even if the magnitude of the natural rate shock is very large.

3.3 Intertemporal Tradeoffs

We now discuss the equilibrium dynamics that prevail once the natural real rate turns positive.

In order to see the implications of carrying out forward guidance, we solve the optimizing IS function forward successively to reach the representation:

$$x_{N+1} = -\sigma \sum_{i=0}^{\infty} (r_{N+1+i} - r_{N+1+i}^n). \tag{7}$$

[12] This characterization does not describe completely the role of the σ parameter; putting aside its role in the expression for $\tilde{\lambda}$, σ enters the linear system as a slope coefficient. But, numerically, it turns out that the predominant contribution of σ to system dynamics comes from its influence on the value of $\tilde{\lambda}$.

We likewise iterate on the Phillips curve and make substitutions to deliver

$$\pi_{N+1} = -\sigma\kappa \sum_{j=0}^{\infty} \beta^i \sum_{i=0}^{\infty} (r_{N+1+i+j} - r^n_{N+1+i+j}). \tag{8}$$

As the preceding expressions indicate, once the natural real rate turns positive, the central bank can carry through the forward guidance policy by setting real interest rates below natural rates for a certain length of time. In addition, equations (7) and (8) imply that the interest sensitivity of aggregate demand matters in determining the magnitude of the gap between real and natural rates implied by a given degree of forward guidance. Specifically, when aggregate demand is more interest elastic, the amount of forward guidance as represented by a given forward guidance vector, $[x_{N+1},\ \pi_{N+1}]$, requires a smaller gap between the real and the natural rates.

But the introduction of forward guidance can stimulate the economy even after the natural rate becomes positive. As a result, the intertemporal cost of forward guidance arises from the possibility that a large degree of forward guidance requires a more sizable and longer-lasting deviation of output from its potential level after the natural rate turns positive. In light of the intertemporal trade-off associated with forward guidance, the Ramsey policymaker must choose the forward guidance vector, $[x_{N+1},\ \pi_{N+1}]$, so as to balance the cost and benefit of the policy. We discuss this issue in the next section.

4 The Optimization of Forward Guidance

We now characterize optimal policy under commitment and discuss how to solve the corresponding optimal policy conditions.

4.1 The Ramsey Policy

We follow Khan, King, and Wolman (2003) in setting out the Ramsey problem in a nonlinear form. A Ramsey policymaker maximizes conditional intertemporal welfare of households from the viewpoint of period zero, subject to specified implementation conditions drawn from the structure of the model. We assume that private employment is subsidized (by the subsidy τ) in a way that extinguishes steady-state effects on the aggregate markup that would otherwise arise from firms' monopolistically competitive character. The optimal resource allocation is therefore attainable at the nonstochastic, zero-inflation steady state. While the markup is subsidized away at the steady state, temporary fluctuations in the markup arise from gradual price adjustment. Their presence implies that firms' rules for setting goods prices become binding constraints on the Ramsey policymaker's optimization. In addition, we augment the implementation constraints with a condition embodying the possibility of a zero lower bound on the short-term nominal interest rate. This becomes a binding constraint in our quantitative analysis when there is a sudden decline in the natural real rate of interest to far below its steady-state value.

We set up the Lagrangian for the optimal policy problem in Table 1. Following Khan, King, and Wolman (2003), we introduce lagged multipliers corresponding to the forward-looking constraints in the initial period, in order to make the problem time invariant (see Table 3 in the appendix). Notice that, if we set the multipliers inherited at period 0 equal to zero, the problem in Table 1 delivers the one in the appendix (Table 3) as a special case.[13]

In order to allow for the presence of the ZLB, we note that the first-order conditions for R_t can be rearranged as follows:

$$\omega_{7t} = \omega_{6t} C_t^{-\sigma^{-1}}$$

where we have made use of the complementarity condition, $\omega_{7t} = \omega_{7t} R_t$. It thus implies that

[13] The stationary reformulation of the Ramsey problem described above for the exact nonlinear optimal policy can be applied to the linear-quadratic optimization problem.

Table 1: *The Lagrangian for the Nonlinear Model*

$$\min_{\{\omega_t\}_{t=0}^{\infty}} \max_{\{d_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \; [(C_t^{1-\sigma^{-1}} - 1)/(1-\sigma^{-1}) \text{ - } \chi_0 H_t^{1+\chi}/(1+\chi)$$

$$+ \; \omega_{1t} \; (A_t H_t/\Delta_t \text{ - } C_t \text{ - } G_t)$$

$$\text{- } \omega_{2t} \; (\alpha\beta\Pi_{t+1}^{\epsilon-1} Z_{1t+1} + A_t H_t C_t^{-\sigma^{-1}}/\Delta_t \text{ - } Z_{1t})$$

$$\text{- } \omega_{3t} \; (\alpha\beta\Pi_{t+1}^{\epsilon} Z_{2t+1} + ((\epsilon-1)\chi_0)/\epsilon(1-\tau_t))(H_t^{1+\chi}/\Delta_t) \text{ - } Z_{2t})$$

$$+ \; \omega_{4t} \; ((1-\alpha)(\tfrac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}} + \alpha\Pi_t^{\epsilon}\Delta_{t-1} \text{ - } \Delta_t)$$

$$\text{- } \omega_{5t} \; (Z_{1t} \; (\tfrac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{1}{1-\epsilon}} \text{ - } Z_{2t})$$

$$+ \; \omega_{6t} \; (C_t^{-\sigma^{-1}} R_t^{-1} \text{ - } \beta \; C_{t+1}^{-\sigma^{-1}} \; \Pi_{t+1}^{-1}) + \omega_{7t}(R_t - 1)]$$

Note: In this table, $d_t = \{ H_t, \Delta_t, C_t, Z_{1t}, Z_{2t}, \Pi_t, R_t \}$ is a vector of decision variables at period t. In addition, $\omega_t = \{ \omega_{1t}, \omega_{2t}, \omega_{3t}, \omega_{4t}, \omega_{5t}, \omega_{6t}, \omega_{7t} \}$ is a vector of Lagrange multipliers chosen in period t. The optimal policy problem in this table solves the Lagrangian given nonstochastic paths of exogenous government consumption, the subsidy rate, and the aggregate productivity shock $\{G_t, A_t, \tau_t\}_{t=0}^{\infty}$ and an initial value of the relative price distortion Δ_{-1}. The steady-state value of τ_t is set to be the one that extinguishes the static monopolistic distortion, so that the nonstochastic steady state with zero inflation corresponds to the efficient allocation.

$R_t = 1$ and $\omega_{7t} > 0$ when the ZLB binds and $R_t > 1$ and $\omega_{7t} = 0$ otherwise. Likewise, we set $R_t = 1$ and $\omega_{6t} > 0$ under the ZLB and $\omega_{6t} = 0$ otherwise because of the household's positive consumption in the above expression.

In the nonstochastic case considered here, the behavior of the model economy under the Ramsey policy can be demarcated into two distinct phases: (1) the initial block of periods where the policy instrument—that is, the short-term nominal interest rate—is at the ZLB; and (2) the subsequent block of periods where the policy rate becomes positive and eventually returns to its steady-state value. During the first phase, the Lagrange multiplier associated with the household Euler equation for consumption (i.e, the IS equation) exceeds zero,

Table 2: *The Lagrangian for the Linear-Quadratic Problem*

$$\min_{\{\phi_t\}_{t=0}^{\infty}} \max_{\{x_t,\pi_t,i_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \text{ [- } (1/2)(\pi_t^2 + \lambda x_t^2)$$

$$\text{- } \phi_{1t} \; (x_t \text{ - } x_{t+1} + \sigma \; (i_t \text{ - } \pi_{t+1} \text{ - } r_t^n))$$

$$\text{- } \phi_{2t} \; (\pi_t \text{ - } \kappa x_t \text{ - } u_t \text{ - } \beta \; \pi_{t+1}) + \phi_{3t} \; i_t \text{]}$$

Note: In this table, $\phi_t = \{ \phi_{1t}, \phi_{2t} \}$ is a vector of Lagrange multipliers chosen at period t. We solve the augmented Lagrangian problems given exogenous deterministic paths of the real natural rate of interest and a vector of initial Lagrange multipliers $\tilde{\omega}_{-1} = \{\phi_{1-1}, \phi_{2-1}\}$.

reflecting the fact that the ZLB constrains the optimal policy rate path. In contrast, this Lagrange multiplier is continuously equal to zero over the second phase, when the ZLB no longer applies.

Thus, for the nonstochastic case, as in Jung, Teranishi, and Watanabe (2005), it is natural to consider a piecewise-linear approximation to the behavior of the model economy. In particular, having obtained the nonlinear optimality conditions, we can compute one linear approximation to the economy for the set of periods $t = 0, ..., N^*$ for which the policy rate $i_t = 0$ and the Lagrange multiplier $\phi_{1,t} > 0$, and another linear approximation for all subsequent periods $t > N^*$ where $i_t > 0$ and $\phi_{1,t} = 0$.

Henceforth, we refer to period $N^{ast} + 1$ as the "lift-off" date; that is, time $t = N^*$ is the final period in which $i_t = 0$ under the optimal policy. We use an iterative "guess-and-verify" method to determine the value of N^{ast}.

4.2 An Equivalence Result

The setting of the subsidy at the efficient level means that the policymaker achieves an efficient nonstochastic steady state with zero inflation. It follows that several Lagrange multipliers are zero at this steady state (i.e., $\omega_2 = \omega_3 = \omega_5 = \omega_6 = 0$).

Lagrange multipliers ω_{2t} and ω_{6t} in the linearized first-order conditions for the optimal policy problem in Table 1 are proportional to Lagrange multipliers in the linear-quadratic problem depicted in Table 2. As shown in the appendix, we have the following relations:

$$\phi_{1t} = \frac{\lambda \sigma^{-1} \beta C^{-\sigma^{-1}-1}}{\sigma^{-1} + \chi} \omega_{6t}; \quad \phi_{2t} = \frac{\lambda}{\kappa} \omega_{2t}$$

where $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(\sigma^{-1}+\chi)}$ and $\lambda = \kappa/\epsilon$. Here, we note that ϕ_{2t} is the Lagrange multiplier associated with the NKPC and ω_{2t} is the Lagrange multiplier associated with one of the nonlinear profit maximization conditions.

The linearized first-order approximations to the nonlinear optimal policy problem in Table 1 and the first-order condition for the linear-quadratic problem in Table 2 can be written as

$$\pi_t = (\frac{\sigma}{\beta})\phi_{1t-1} - (\phi_{2t} - \phi_{2t-1}) \tag{9}$$

$$\lambda x_t = \kappa \phi_{2t} - (\phi_{1t} - \beta^{-1} \phi_{1t-1}). \tag{10}$$

where $\phi_{1-1} = \phi_{2-1} = 0$. and the steady-state share of government consumption in output is zero.[14]

5 Quantifying the Limitations of Forward Guidance

To quantify the benefits and limitations of forward guidance at the ZLB, we examine scenarios in which an exogenous decline in aggregate demand pushes the natural real rate of interest below zero. which, in turn, prompts policymakers to cut the short-term nominal interest rate to zero. In this section, the values assigned to the structural parameters are modeled after Woodford (2003): $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$.[15] We consider two specifications

[14]In related equivalence results, Benigno and Woodford (2005, 2008) show that the equivalence holds for the optimal policies from the timeless perspective; while Levine, Pearlman and Pierse (2008) analyze linear-quadratic approximations to the Ramsey policy. We have here produced an analogue, under ZLB conditions, to their result. We have restricted ourselves to the case where the nonstochastic steady state is efficient, and focused on perfect-foresight equilibrium dynamics. This limitation is dictated by our piecewise-linear approximation of the model equilibrium.

[15]This compares with Woodford's $\beta = 0.99$, $\kappa = 0.024$, and $\sigma = 0.157^{-1}$. It should be noted that Rotemberg and Woodford (1997) and Woodford (2003) used the symbol σ to denote the degree of relative risk aversion, whereas we follow the notation of Eggertsson and Woodford (2003) in using σ to denote the interest elasticity, that is, the inverse of the risk aversion parameter. Thus, the value of $\sigma = 6$ used in this paper corresponds to the risk aversion parameter estimate of 0.157 obtained by Rotemberg and Woodford (1997) and used in the benchmark parameterizations of Woodford (2003).

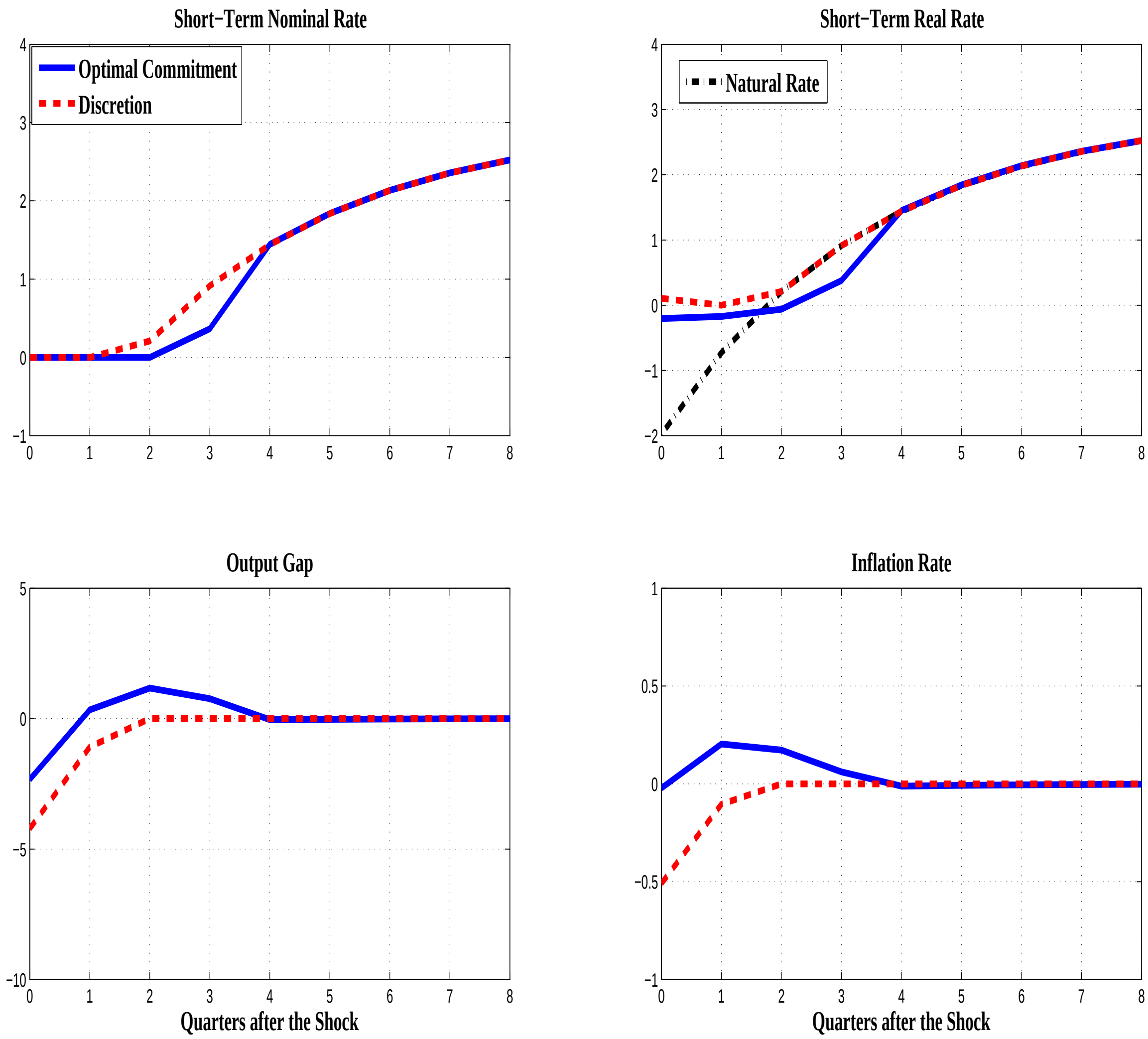
for the natural rate shock: an autoregressive shock of modest size and persistence—as in Adam and Billi (2003, 2006) and Nakov (2008)—that may be viewed as characteristic of the "Great Moderation" era; and a more severe and persistent shock that can be interpreted as representing a "Great Recession"-style episode. Section 7 will consider alternative values of the interest elasticity parameter (σ) and will examine the case in which the shock follows a two-state Markov process, as in Eggertsson and Woodford (2003).

In Figure 4 we consider a "Great Moderation"-style shock of about 5 percent to the natural real interest rate, with the shock fading out within a few quarters; specifically, the shock follows an AR(1) process with parameter $\rho = 0.75$. As shown in the upper-left panel, the optimal policy under discretion is only constrained by the ZLB for two quarters, whereas the optimal policy under commitment keeps the short-term nominal interest rate at zero for an additional quarter. By keeping the nominal interest rate at zero for a somewhat longer period than under discretion, the optimal commitment dampens the impact of the natural rate shock on the output gap. The deviations of inflation from zero are consistently mild, and these deviations cumulate to about zero over a couple of years, so that the inflation path closely resembles that implied by a simple rule with a constant price level target.

It is, however, the case of a large AR(1) shock, shown in Figure 5, that brings home the limitations of forward guidance. Here, even promises of fluctuations in inflation for several additional quarters fails to prevent the natural rate shock from causing a deep output gap at time zero. Moreover, a substantial rise in inflation is required to push down real interest rates and thereby avoid an even steeper decline in output such as that observed under the discretionary policy.

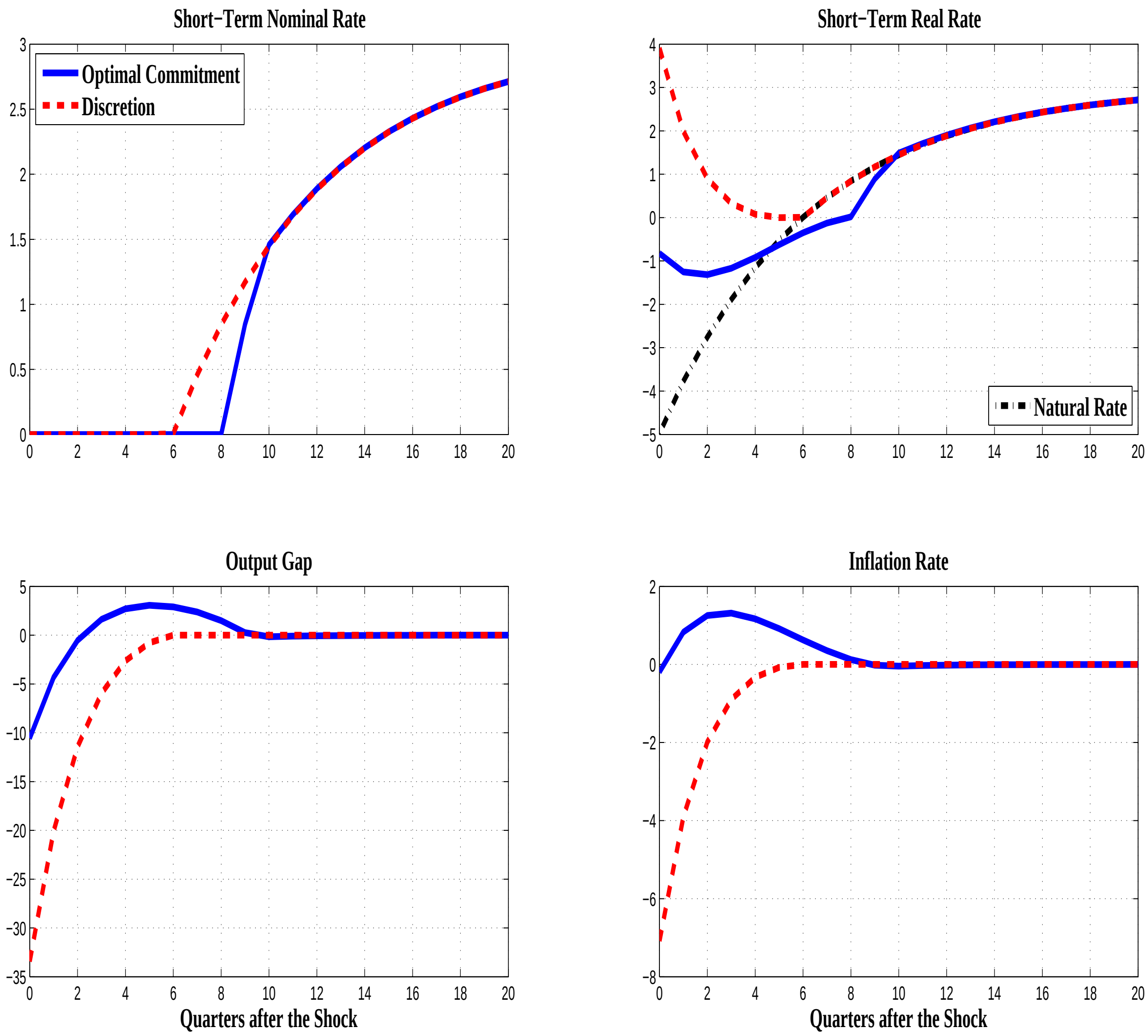
For each of these two shocks, Figure 6 depicts the corresponding trajectories for the Lagrange multiplier on the dynamic IS equation. As discussed in Section 4.1, this Lagrange multiplier is positive over the periods where the ZLB is an active constraint and falls to zero once the ZLB no longer constrains the optimal setting of the short-term nominal interest rate. Thus, the magnitude of this Lagrange multiplier provides a useful measure of the extent to which the ZLB reduces social welfare at each point in time. For the "Great Moderation"-style shock, these welfare costs are very small and transitory, reflecting the effectiveness of forward guidance in providing stabilization outcomes that are nearly as good as in the absence of the ZLB. In contrast, the Lagrange multiplier is roughly an order of magnitude

Figure 4: "Great Moderation"-Style Shock



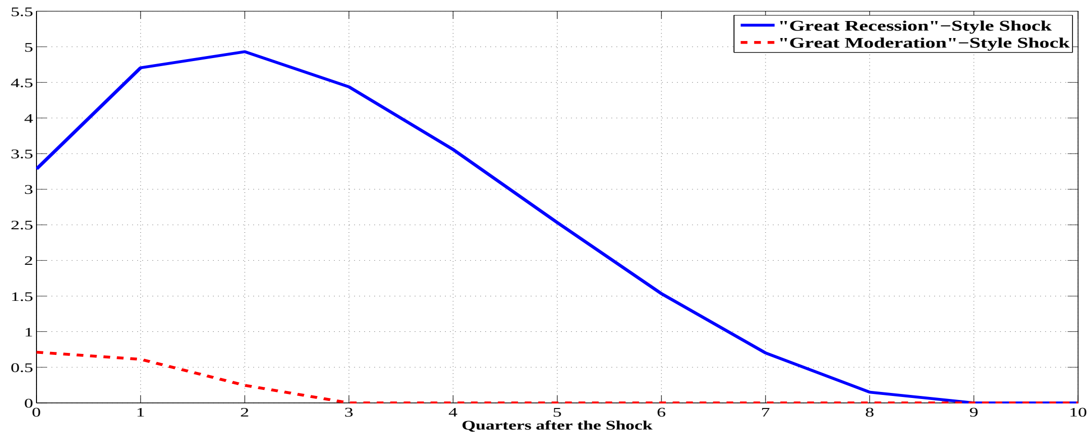
Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate shock follows an AR(1) process with first-order autocorrelation coefficient $\rho = 0.75$. The short-term nominal interest rate, the short-term real interest rate, and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points.

Figure 5: "Great Recession"-Style Shock



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate shock follows an AR(1) process with first-order autocorrelation coefficient $\rho = 0.85$. The short-term nominal interest rate, the short-term real interest rate, and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points.

Figure 6: The Lagrange Multiplier on the Dynamic IS Equation



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate follows an AR(1) process with first-order autocorrelation coefficient $\rho = 0.75$ for the "Great Moderation"-style shock and $\rho = 0.85$ for the "Great Recession"-style shock. The Lagrange multiplier is depicted using a scaling factor of 10^{-5}.

larger—and much more persistent—for the "Great Recession"-style shock, thereby providing further perspective on the limitations of forward guidance in this case.

6 Constant Price-Level Targeting

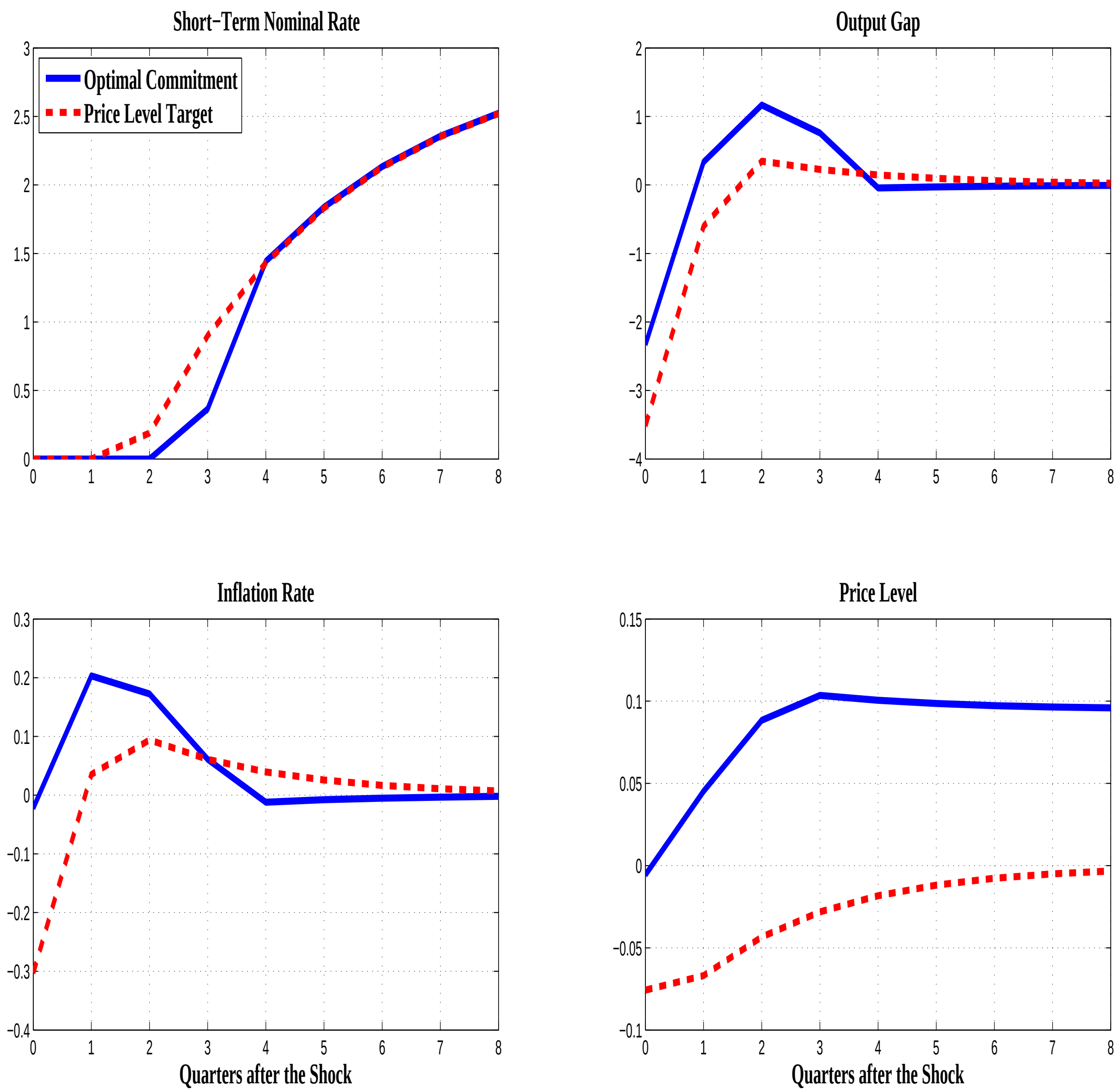
Now we consider the extent to which the optimal commitment can be replicated by a constant price-level targeting rule, such as the one proposed by Eggertsson and Woodford (2003):

$$p_t + \frac{\lambda}{\kappa} x_t = p* \tag{11}$$

where p_t denotes the logarithm of the price level and $p*$ denotes the target value (which is a time-invariant constant) for a linear combination of log prices and the output gap—i.e., for the "output-gap adjusted price index," in the terminology of Eggertsson and Woodford (2003). As noted by Eggertsson and Woodford (2003), this rule involves no special provisos related to the ZLB, and hence might be simpler to communicate than the optimal commitment.

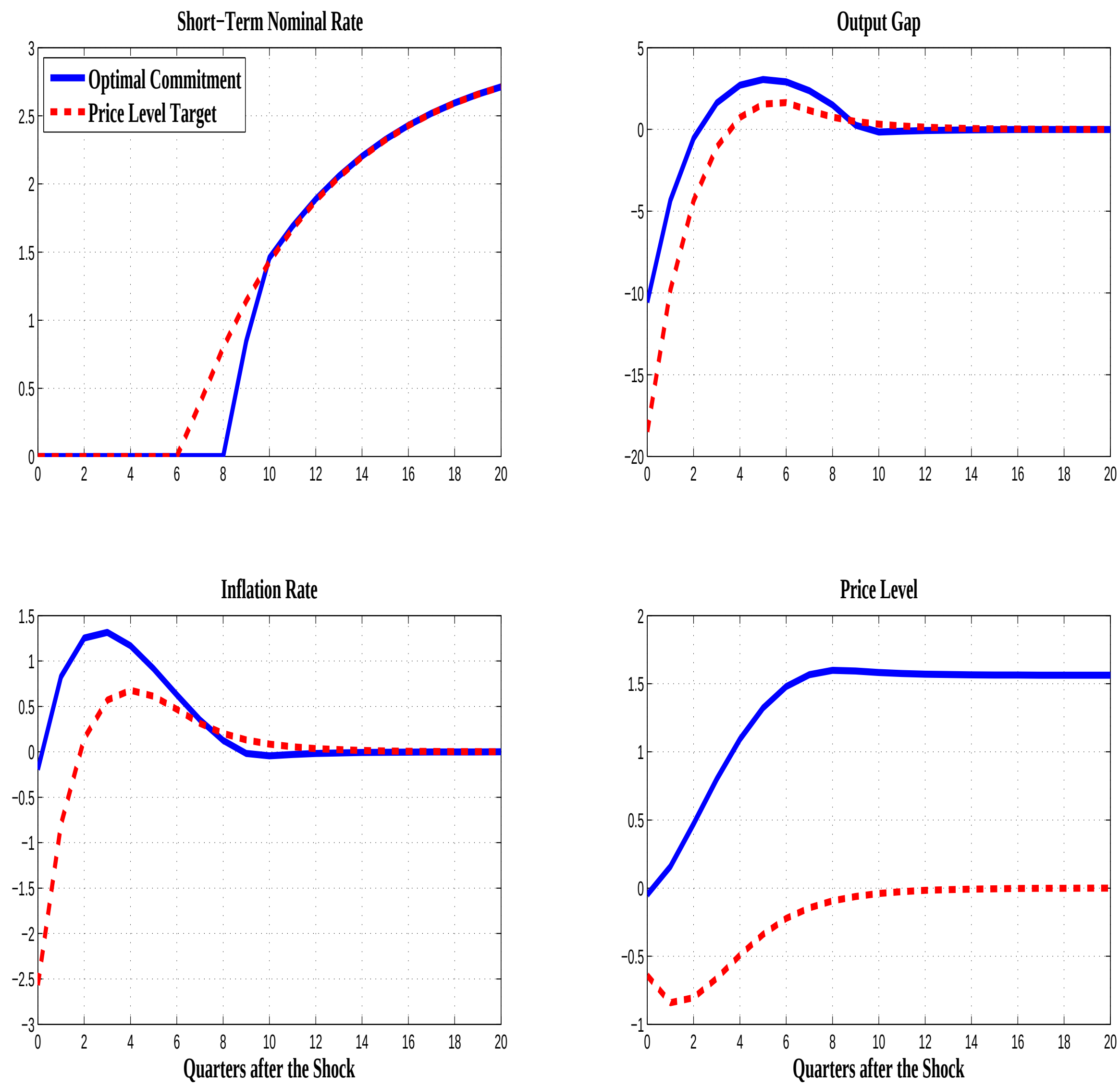
Figure 7 depicts the performance of the constant price-level targeting rule for the "Great

Figure 7: "Great Moderation"-Style Shock with a Constant Price-Level Targeting Rule



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The "price level targeting " case refers to a rule that targets the output gap-adjusted price level. The natural rate shock follows an AR(1) process with first-order autocorrelation coefficient $\rho = 0.75$. The short-term nominal interest rate and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points; and the price level is expressed as the percent deviation from the initial price level prior to the onset of the shock.

Figure 8: “Great Recession”-Style Shock with a Constant Price-Level Targeting Rule



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The “price level targeting ” case refers to a rule that targets the output gap-adjusted price level. The natural rate shock follows an AR(1) process with first-order autocorrelation coefficient $\rho = 0.85$. The short-term nominal interest rate and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points; and the price level is expressed as the percent deviation from the initial price level prior to the onset of the shock.

Moderation"-style shock. In this case, the simple rule generates macroeconomic outcomes that are nearly as good as those obtained under the optimal commitment; that is, the output gap and inflation rate exhibit only slightly increased variability.

As shown in Figure 8, however, the stabilization performance of the constant price-level targeting rule in response to the "Great Recession"-style shock is decidedly inferior to that of the commitment policy. In this case, the optimal policy prescribes a persistently elevated inflation rate that pushes down the *ex ante* real interest rate and thereby dampens the initial impact of the shock. In contrast, the constant price level targeting rule generates an initial phase of deflation—with a much steeper drop in output than under the optimal policy—and a subsequent phase of positive inflation that eventually brings the price level back to target.

Let us consider why the size of the natural rate shock affects the extent to which constant price-level targeting replicates the optimal commitment. By integrating the optimal policy condition (9) forward from period 0 onward, we obtain the following expression:

$$p_\infty = p_{-1} + \frac{\sigma}{\beta} \sum_{t=0}^{N^*} \phi_{1t} \tag{12}$$

where p_∞ denotes the eventual price level at time $t = \infty$; p_{-1} denotes the initial price level just prior to the onset of the natural rate shock; ϕ_{1t} is the Lagrange multiplier on the IS equation; and N^* is the final period in which the nominal interest rate i_t remains at the ZLB under the optimal policy.[16] As discussed in Section 4, the Lagrange multiplier ϕ_{1t} is positive for $t = 0, \cdots, N^*$ and zero for $t > N^*$.

Evidently, the optimal policy under commitment implies upward base drift whenever the central bank is constrained by the ZLB, and optimal degree of base drift is directly proportional to the severity of this constraint, as measured by the sum of Lagrange multipliers over periods $t = 0, \cdots, N^*$. As we have seen in Figure (6), these Lagrange multipliers are quite small for a "Great Moderation"-style shock but are much larger and more persistent in the case of a "Great Recession"-style shock. Hence, it is not surprising that the constant price-level targeting rule performs reasonably well in the former case but very poorly in the latter case.

[16]This expression incorporates the initial conditions $\phi_{1t} = \phi_{2t} = 0$ for $t = -1$, which reflect the Pareto optimality of the non-stochastic steady state.

7 Sensitivity Analysis

In this section, we perform sensitivity analysis by varying the interest elasticity of aggregate demand, and we examine the case in which the natural rate shock follows a two-state Markov process, as in Eggertsson and Woodford (2003), rather than an AR(1) process.

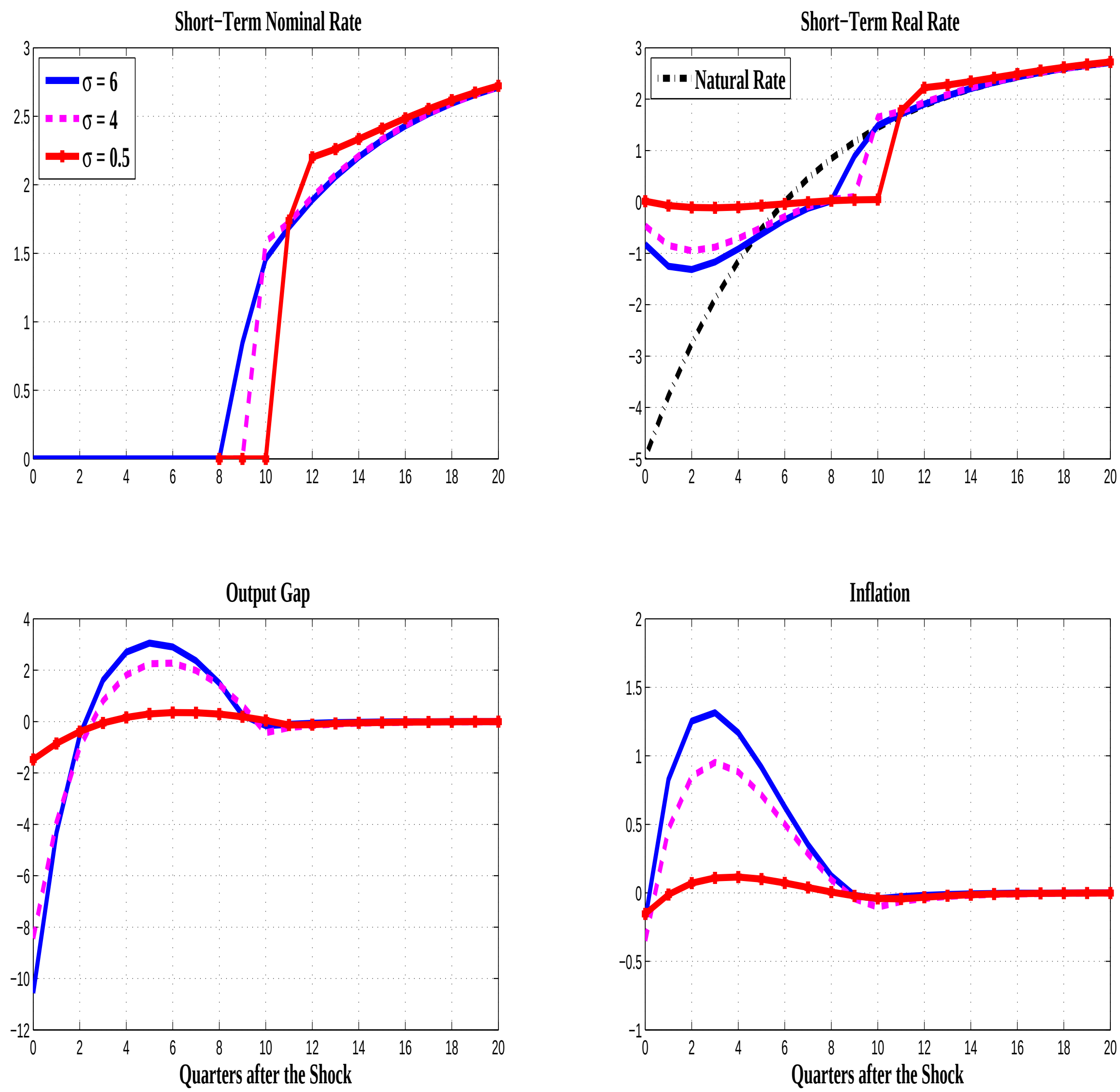
7.1 The Interest Elasticity of Aggregate Demand

In the foregoing analysis, we have used the benchmark parameterization of Woodford (2003), in which the interest elasticity parameter was approximately $\sigma = 6$. Rotemberg and Woodford (1997) obtained this parameter value by applying minimum-distance estimation to a small stylized New Keynesian model, using U.S. aggregate time series data (that is, real GDP, the GDP price inflation rate, and the federal funds rate). Of course, numerous other studies have estimated the slope of the dynamic IS equation using a variety of empirical procedures. For example, Amato and Laubach (2003) obtained the estimate $\sigma = 4$ using essentially the same estimation procedure as Rotemberg and Woodford (1997) but allowing for nominal rigidity in wages as well as prices. In contrast, Eggertsson and Woodford (2003) specified $\sigma = 0.5$ and noted that this value "represents a relatively low degree of interest sensitivity of aggregate expenditure."[17]

Figure 9 depicts the optimal policy path and associated macroeconomic outcomes for different values of the parameter σ for the "Great Recession"-style shock that was considered in Sections 5 and 6. Evidently, the shortcomings of forward guidance are roughly similar for either widely used value of the interest elasticity ($\sigma = 4$ or $\sigma = 6$). In contrast, when the interest elasticity is made much lower ($\sigma = 0.5$), the commitment policy delivers virtually immaculate stabilization outcomes that are roughly comparable to the results obtained by Eggertsson and Woodford (2003).

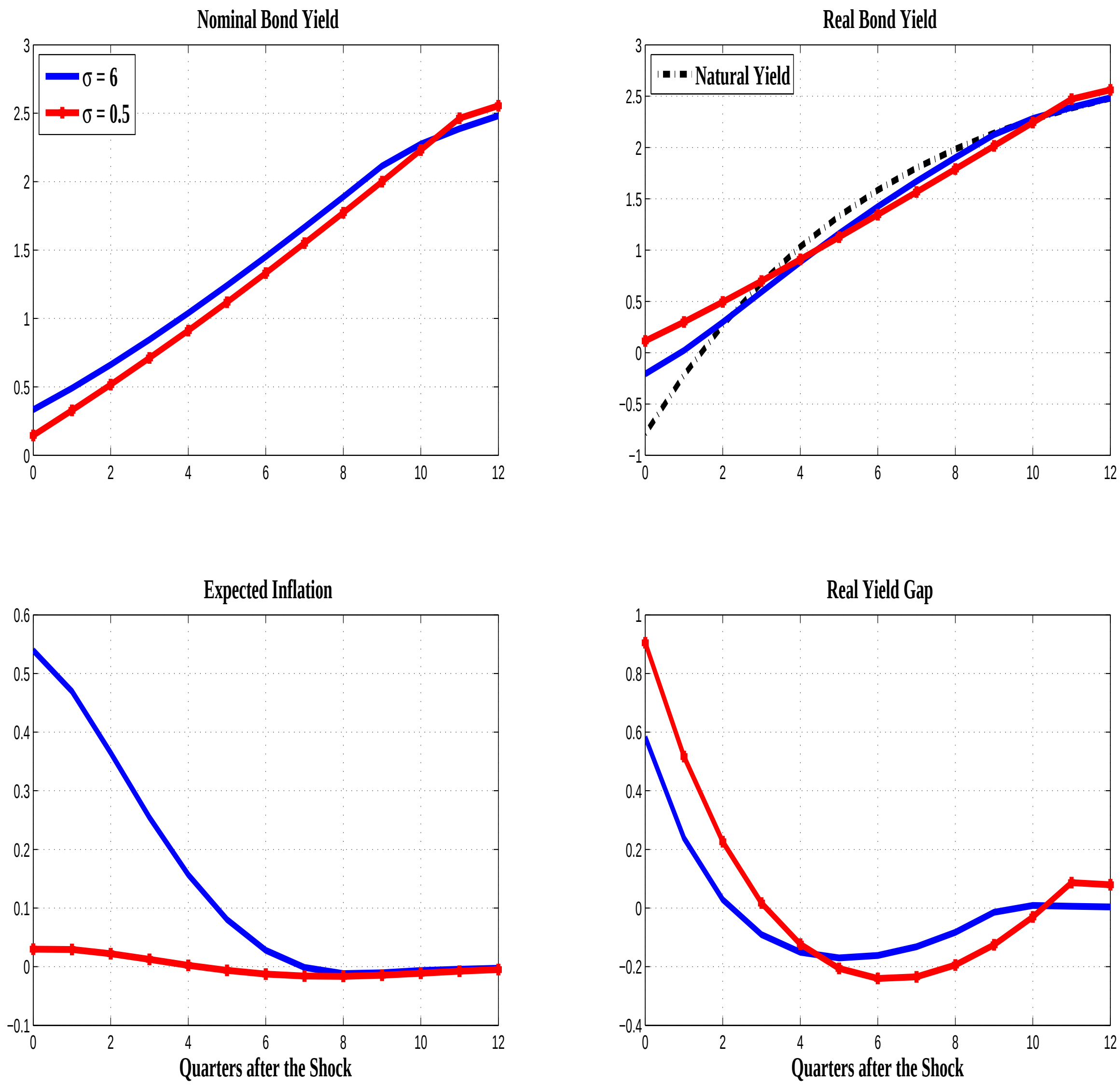
[17]Footnote 36 of Eggertsson and Woodford (2003) explains their choice as follows: "We prefer to bias our assumptions in the direction of only a modest effect of interest rates on the timing of expenditure, so as not to exaggerate the size of the output contraction that is predicted to result from an inability to lower interest rates when the zero bound binds." The idea that aggregate demand may become markedly less interest-sensitive at the ZLB is certainly intriguing and deserves further empirical investigation. Nonetheless, we have conducted some preliminary analysis (not reported here) regarding the policy implications of such a mechanism, using a stylized Markov regime-switching model where the interest elasticity of aggregate demand shifts at the same time as the natural real interest rate, and we have found that forward guidance is *even less effective* than in our benchmark specification of $\sigma = 6$.

Figure 9: The Interest Elasticity of Aggregate Demand



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$ and $\kappa = 0.024$, with three alternative values of σ (0.5, 1, and 6). The natural rate shock follows an AR(1) process with a persistence parameter of 0.875.

Figure 10: Longer-Term Yields and Expected Inflation



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$ and $\kappa = 0.024$.

Figure 10 depicts how the value of σ influences the behavior of bond yields and expected inflation for securities with a 12-quarter maturity (This maturity was chosen to be sufficiently long to span the entire episode of the "Great Recession"-style shock.)

7.2 Two-State Markov Shocks

The analysis of Eggertsson and Woodford (2003) focused on natural rate shocks generated by a two-state Markov process. To gauge the implications of this specification, let us now consider a Markov shock with magnitude and persistence comparable to the "Great Recession"-style AR(1) shock that we have been considering heretofore. In particular, we shall assume that the Markov shock reduces the natural real rate by about 4 percent and lasts for seven quarters.

As shown in Figure 11, the implications of σ are roughly similar for the case of a Markov shock as for the case of an AR(1) shock. In particular, the optimal commitment generates virtually immaculate stabilization outcomes when $\sigma = 0.5$—the parameter value used by Eggertson and Woodford (2003)—but are much less appealing for commonly-used values of this parameter ($\sigma = 4$ or $\sigma = 6$). Although not shown here, the considerations related to a constant price-level targeting rule are also essentially the same for a two-state Markov shock as for an AR(1) shock. The crucial issue for stabilization performance is *not* the shape of the shock trajectory, but the magnitude and persistence of the natural rate shock (that is, drawn from the "Great Moderation" era, as opposed to representing a "Great Recession"-style event) and the interest sensitivity of aggregate demand.

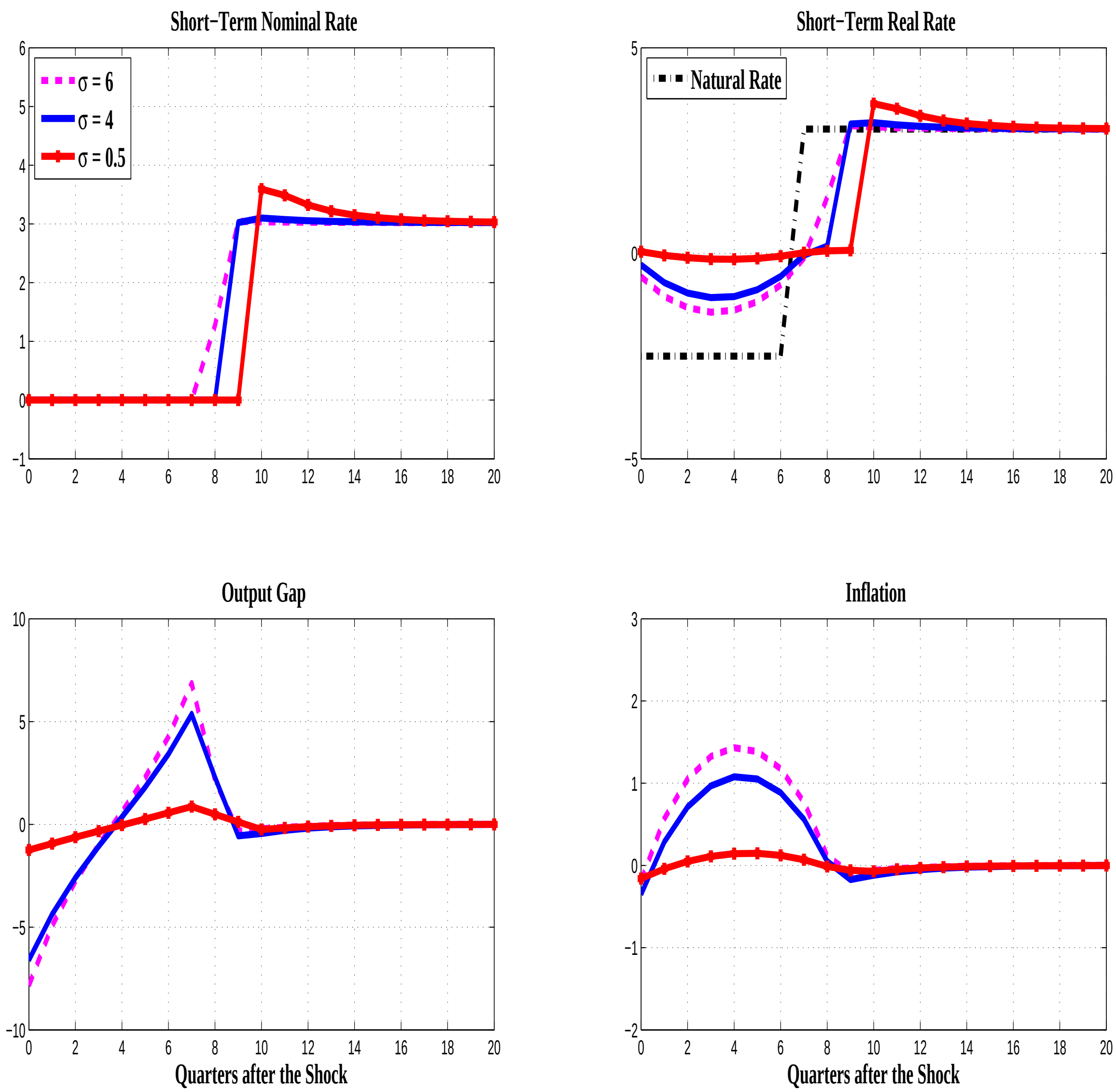
8 Implications of Uncertainty

Up to this point, we have analyzed cases where all agents have perfect foresight regarding the path of the natural real interest rate from period zero onward. In this section, we consider the implications of uncertainty about the future path of natural rates.

8.1 Two-State Markov Process

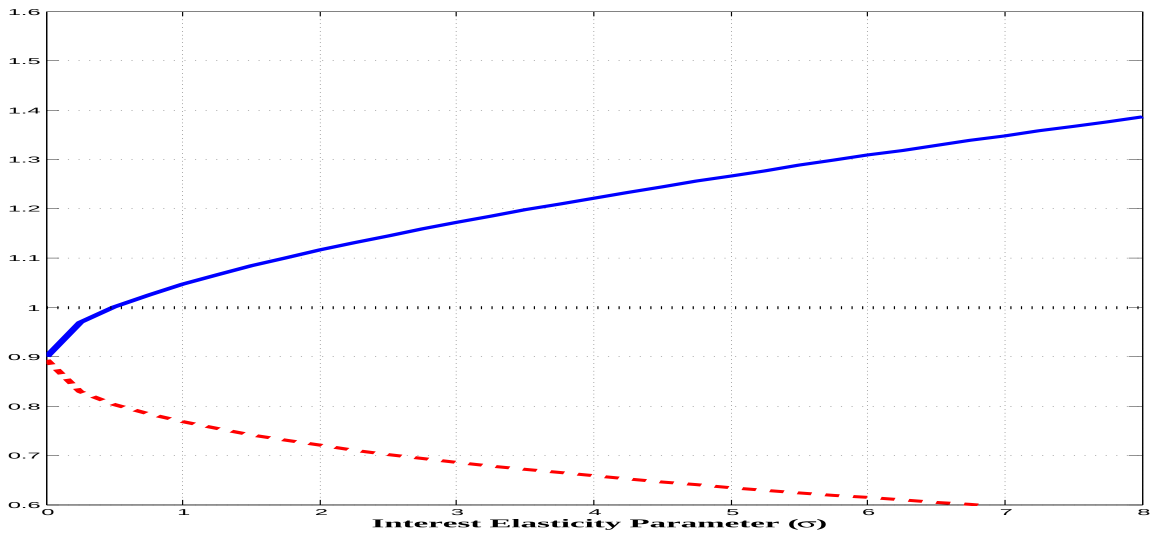
Eggertsson and Woodford (2003) consider experiments where the two-state Markov process for the natural rate of interest is stochastic, in the sense that there is uncertainty regarding

Figure 11: The Interest Elasticity and Markov Shocks



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$ and $\kappa = 0.024$, with three alternative values of σ (0.5, 1, and 6). The natural rate shock follows a two-state Markov process.

Figure 12: The Mechanics of Forward Guidance with Two-State Markov Shock Process



Note: This figure depicts how the interest elasticity parameter σ influences the magnitudes of the two eigenvalues that determine the behavior of the dynamic system through the period over which the policy rate is held at zero. The solid and dashed lines represent the eigenvalues of matrix $\tilde{A}$. In this two-value Markov case, we use the baseline parameterization of $\beta = 0.9925$ and $\kappa = 0.024$ with $\xi = 0.1$.

the period at which the natural real interest rate reverts to its steady-state value. As they show, this form of uncertainty actually magnifies the gains from forward guidance.

In this setting, the natural real rate r_t^n has only two possible states $\underline{r}^n$ and $\bar{r}^n$, where $\underline{r}^n < 0$ and $\bar{r}^n > 0$. The natural rate starts out at the negative state (that is, $r_0^n = \underline{r}^n$), and the timing of its return to steady state follows a truncated Markov process. In particular, for periods $t = 1, \cdots, \tilde{N}$, this process satisfies the following Markov transition matrix:

$$\begin{pmatrix} 1-\xi & \xi \\ 0 & 1 \end{pmatrix} \tag{13}$$

where $1-\xi$ is the probability that the natural rate remains at its negative value ($\underline{r}^n$), and ξ is the probability that the natural rate reverts to its steady state value ($\bar{r}^n$). This Markov process is *truncated* because the natural rate surely reverts to steady state at time $t = \tilde{N}+1$ if it has not already done so at some previous date $1 \leq t \leq \tilde{N}$. Thus, this process encompasses perfect foresight as the special case in which $\xi = 0$; in that case, the natural rate remains at its negative state with certainty for periods $t = 0, \cdots, \tilde{N}$ and then reverts to steady state with certainty at time $t = \tilde{N} + 1$.

Now consider the following expressions for inflation and the output gap at any given period t for which the natural rate is still negative:

$$\begin{pmatrix} \tilde{\pi}_t \\ \tilde{x}_t \end{pmatrix} = (1-\xi)A \begin{pmatrix} \tilde{\pi}_{t+1} \\ \tilde{x}_{t+1} \end{pmatrix} + \xi A \begin{pmatrix} \hat{\pi}_{t+1} \\ \hat{x}_{t+1} \end{pmatrix} + \sigma \begin{pmatrix} \kappa \\ 1 \end{pmatrix} \underline{r}, \tag{14}$$

where $\tilde{\pi}_t$ and $\tilde{x}_t$ denote the inflation rate and the output gap in period t conditional on the negative natural rate at period t, while $\hat{\pi}_{t+1}$ and $\hat{x}_{t+1}$ are the inflation rate and output gap values that would occur if positive natural rates prevail from period $t+1$ onward.

We now iterate equation (14) forward to yield the outcomes for inflation and the output gap as a function of the expected future values of these variables.

$$\begin{pmatrix} \tilde{\pi}_{\tilde{N}-j} \\ \tilde{x}_{\tilde{N}-j} \end{pmatrix} = \tilde{A}^j A \begin{pmatrix} \hat{\pi}_{\tilde{N}+1} \\ \hat{x}_{\tilde{N}+1} \end{pmatrix} + \xi \sum_{i=1}^{j} \tilde{A}^{j-i} A \begin{pmatrix} \hat{\pi}_{\tilde{N}-i+1} \\ \hat{x}_{\tilde{N}-i+1} \end{pmatrix} + \sigma (\sum_{i=0}^{j} \tilde{A}^i) \begin{pmatrix} \kappa \\ 1 \end{pmatrix} \underline{r}^n \tag{15}$$

for $j = 0, \cdots, \tilde{N}$, and the matrix $\tilde{A}$ is defined as follows:

$$\tilde{A} = (1-\xi)A \tag{16}$$

Compared with the perfect-foresight AR(1) case, the introduction of uncertainty makes it possible for the central bank to make state-contingent commitments on future inflations and output gaps. Thus, multiple forward guidance vectors should be included in equation (15) as opposed to the perfect foresight AR(1) case. However, the economy still behaves almost like an "uncontrolled" system—which is to say, one with a tendency to diverge instead of exhibiting dynamic stability—and depends on the vectors $\{[x_{\tilde{N}-i+1}, \pi_{\tilde{N}-i+1}]\}_{i=0}^{\tilde{N}}$, the exogenous path of the natural rate, and the matrix $tildeA$. It should be noted that the dynamic system specified in (15) converges to the system (6) as ξ approaches one.

Figure 12 depicts the eigenvalues of the matrix $\tilde{A}$ as a function of the interest elasticity parameter σ. By comparison with Figure 3, it is evident that the eigenvalues in the stochastic case are *lower* than in the case of perfect foresight. Moreover, the parameter ξ (which determines the average duration over which the natural rate remains negative) and the parameter σ are each crucial in determining the eigenvalues of $\tilde{A}$ and hence the dynamic stability of the output gap and inflation rate over the periods when policy is constrained by the ZLB.

Evidently, forward guidance can provide more appealing stabilization outcomes in the case of a truncated Markov process environment than in the case of perfect foresight. This result may seem somewhat surprising, since the introduction of uncertainty might be expected to reduce the effectiveness of forward guidance. However, this result depends crucially on the fundamental nature of a two-state Markov process, namely, that the natural rate reverts *instantly* to its steady-state value. Under perfect foresight, as in the preceding sections, policymakers and private agents know that this recovery will not occur until a later date, and the economy remains uncontrolled until that time; that is, forward guidance only works "at a distance." In contrast, in the stochastic case, a complete recovery may occur at every date, and hence the stimulus from forward guidance is much less distant–indeed, perhaps just one period in the future–and hence much more effective in stabilizing the output gap and inflation.

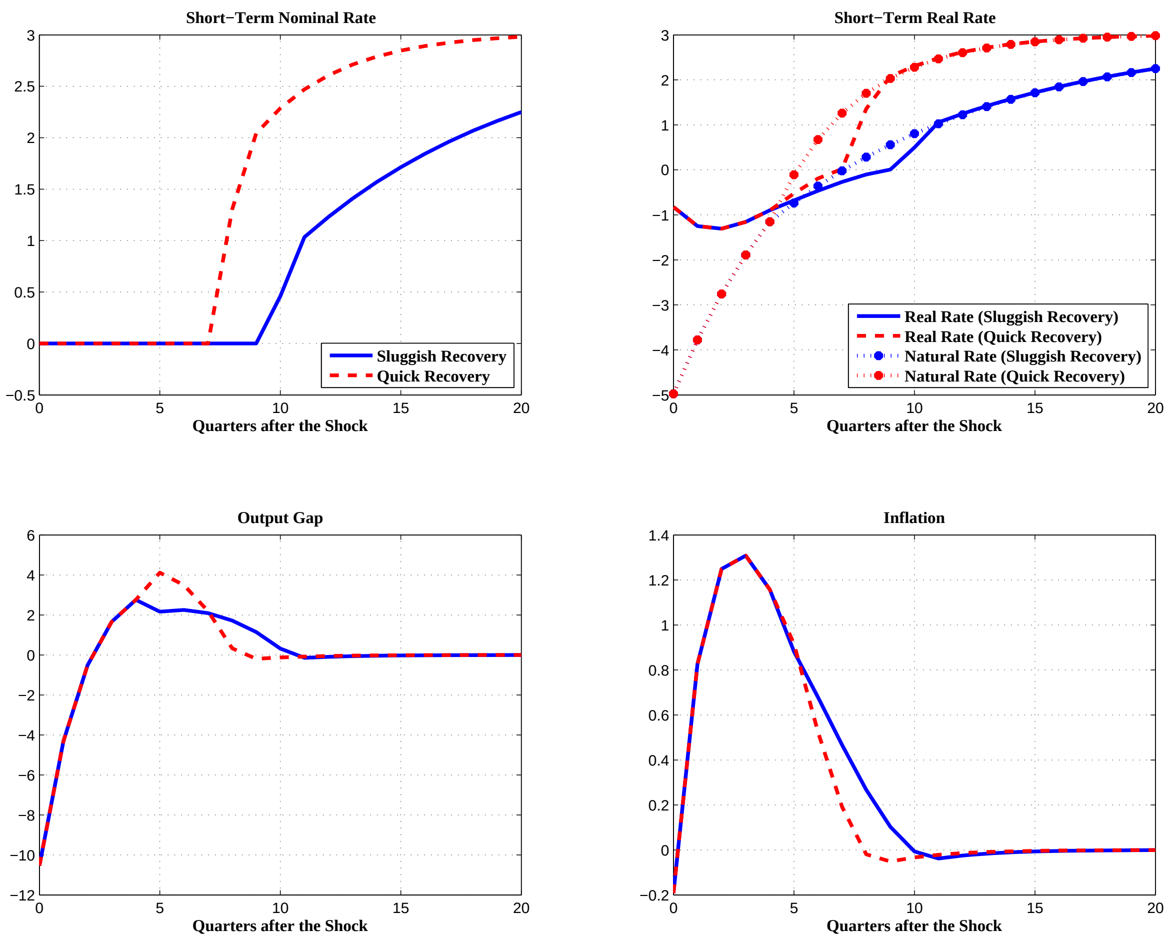
This analysis is also helpful in understanding why forward guidance may provide very effective stabilization performance even if the ZLB is likely to be a binding constraint for many periods. For example, with $\sigma = 0.5$ and $\xi = 0.1$ (as in the calibration of Eggertsson and Woodford, 2003), both eigenvalues of the matrix $\tilde{A}$ are inside the unit circle. Thus, for that calibration, the optimal policy can obtain near-zero outcomes for the output gap and inflation rate by promising a small state-contingent stimulus at each possible liftoff date and thereby offsetting the contractionary impact of the negative natural real rate.

Nonetheless, it is worth noting that such implications may be quite sensitive to modest variations in the parameter values. For example, if $\sigma = 1$ and $\xi = 0.05$, then one of the eigenvalues of $\tilde{A}$ lies outside the unit circle. Moreover, since the expected duration of the shock is 20 quarters, the trajectories for the output gap and inflation rate place substantial weight on relatively high powers of $\tilde{A}$, and hence these outcomes may be very unappealing.

8.2 Shifts in the Autoregressive Coefficient

Let us now consider an alternative means of modeling uncertainty in the duration of unusual economic conditions. We specify the natural rate shock as AR(1) and have uncertainty refer to the rate at which normal conditions resume; that is, to uncertainty about the AR(1) coefficient that propagates the shock. In contrast to the case of a stochastic Markov process, we find that the optimal policy under commitment and the expected outcomes for the output gap and inflation rate are fairly similar to the case of perfect foresight.

Figure 13: Stochastic Shifts in the Recovery of Aggregate Demand



Note: This simulation was performed using the baseline parameterization, $\beta = 0.9925$, $\kappa = 0.024$, $\sigma = 6$. The persistence parameter for the natural rate shock is either $\rho_1 = 0.90$ or $\rho_2 = 0.75$ after period 5 onward, while it is $\rho_0 = 0.85$ by the end of period 4.

We assume that the natural rate follows an AR(1) process with the possibility of a shift in the autoregressive parameter. A negative shock to the natural real rate of interest strikes in period 0 and decays in each period at a constant rate ρ_0 through period 4:

$$r_0^n = r^n + \rho_0^t \epsilon_0$$

where we set $\rho_0 = 0.85$. From period 5 onward, the AR parameter of the natural rate shock process is subject to an exogenous permanent shift, assuming a value of either $\rho_1 = 0.90$ or $\rho_2 = 0.75$ with a probability of 0.50 for each case.

$$\rho_t = \begin{cases} \rho_1 = 0.90 & \text{with } p = 0.5 \\ \rho_2 = 0.75 & \text{with } p = 0.5 \end{cases}$$

Hence, the timing of shifts in the persistence parameter is perfectly known at period 0 but the size of the change in the persistence parameter is not known until period 5. Conditional on this process, we use log-linearized optimal policy conditions in order to compute the period in which the policy rate is raised above zero. Our motivation for this type of shift in the decay rate for the natural rate shock is to analyze the impact of changes in the expected pace of recovery on the optimal commitment for forward guidance; for example, the central bank would presumably rely less on the stimulus from forward guidance when the economy is expected to return to normal situations rapidly.

Figure 13 shows that the optimal commitment requires that the central bank make state-contingent announcements regarding the timing of the departure from the zero lower bound. For example, a downward shift in the persistence parameter is associated with an earlier departure from zero policy rates; that is, the policy rate remains at zero for 7 quarters when the persistence parameter declines from $\rho_0 = 0.85$ to $\rho_2 = 0.75$ at period 5, whereas the policy rate stays at zero for 9 quarters when the persistence parameter increases from $\rho_0 = 0.85$ to $\rho_2 = 0.90$ at period 4.

9 Conclusions

Our analysis in this paper has reconsidered optimal monetary policy in a New Keynesian model under conditions where a natural rate shock makes the zero lower bound on nominal interest rates a binding constraint on monetary policy. The existing literature has generally considered shocks of a size that are plausible in considering most countries' experience from 1984 to 2007, but are too low for thinking about the shock that preceded the 2008 downturn. However, when we consider a relatively large and persistent shock to the natural rate, we find that forward guidance alone is not sufficient to keep output close to potential and inflation close to the long-run goal. These results suggest that there could indeed be a role for nontraditional monetary policies as a complement to forward guidance. Thus, incorporating such mechanisms into a New Keynesian modeling framework is an important direction for further research.

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Appendix: The Optimal Policy Problem

The model economy corresponds to the baseline New Keynesian model and includes the staggered price-setting model of Calvo (1983), with consent given in any period only to a fraction of firms $(1-\alpha)$ to alter goods prices. These intermediate-goods producing firms generate differentiated goods (indexed by z) using the technology $Y_t(z) = A_t H_t(z)$ where $Y_t(z)$ is output, A_t the productivity shock that all firms face, and $H_t(z)$ firm z's per-period hours input. The rest of the model, including aggregation across goods through the Dixit-Stiglitz apparatus, follows similarly standard lines (see e.g., King, 2000, or Woodford, 2003). We note, however, that relative price distortions mean that aggregate output is equal to $\frac{A_t}{\Delta_t} H_t$, Δ_t being a per-period relative price distortion term, rather than $A_t H_t$; see Khan, King, and Wolman (2003). Following these authors, we set up the Lagrangian problem for aggregate welfare maximization as:

$$\pounds = \frac{C_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} - \frac{\chi_0 H_t^{1+\chi}}{1+\chi} + \tag{17}$$

$$\omega_{1t}\left[\frac{A_t}{\Delta_t} H_t - C_t - G_t\right] \tag{18}$$

$$-\omega_{2t}\left[\frac{A_t H_t}{\Delta_t C_t^{\sigma^{-1}}} + \alpha\beta\Pi_{t+1}^{\epsilon-1} Z_{1t+1} - Z_{1t}\right] \tag{19}$$

$$-\omega_{3t}\left[\frac{\chi_0 H_t^{1+\chi}}{(1-\tau_t)\left(1-\epsilon^{-1}\right)\Delta_t} + \alpha\beta\Pi_{t+1}^{\epsilon} Z_{2t+1} - Z_{2t}\right] \tag{20}$$

$$+\omega_{4t}\left[(1-\alpha)\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} + \alpha\Pi_t^{\epsilon}\Delta_{t-1} - \Delta_t\right] \tag{21}$$

$$-\omega_{5t}\left[Z_{1t}\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{1-\epsilon}} - Z_{2t}\right] \tag{22}$$

$$+\omega_{6t}\left[C_t^{-\sigma^{-1}} R_t^{-1} - \beta C_{t+1}^{-\sigma^{-1}} \Pi_{t+1}^{-1}\right] + \omega_{7t}(R_t - 1) \tag{23}$$

We have assumed separable preferences for the representative household, parameterized by $\frac{1}{\sigma}$ (the relative risk-aversion coefficient), $\frac{1}{\chi}$ (Frisch labor supply elasticity), and β (the discount factor). C_t is the household's period t level of consumption and H_t is the representative household's total hours worked. Variables ω_{1t} to ω_{7t} respectively denote the Lagrange multipliers attached to the implementability constraints (18)-(23). Expression (18) is the economy's resource constraint.

It eases the analysis to cast firms' problem in recursive form. The recursive representation of profit maximization appears in constraints (19), (20), and (22). Π_t is the gross inflation rate for aggregate output, and the parameter ϵ is the demand elasticity for a typical firm's product. The relative price distortion, Δ_t, is represented by constraint (21) The household's Euler equation for consumption constitutes expression (23); under ZLB, this constraint incorporates a unit value for the gross nominal interest rate ($R_t = 1$). More specifically, we have $R_t = 1$ and $\omega_{7t} > 0$ under ZLB, while $R_t > 1$ and $\omega_{7t} = 0$ in the absence of ZLB.

Finally, A_t and G_t denote shocks to productivity and government purchases, respectively. They follow AR(1) processes:

$$\log A_t = \rho_A \log A_{t-1} + \epsilon_{At} \tag{24}$$

$$\log G_t = \rho_G \log G_{t-1} + \epsilon_{Gt} \tag{25}$$

where ϵ_{At} and ϵ_{Gt} are all i.i.d. normal, mean-zero innovations. These two exogenous shocks are sources of variation in the natural rate of interest, which becomes the trigger, in our experiments in Section 4, for the economy's reaching a zero lower bound for the nominal interest rate. Our derivations in this paper, however, pertain to perfect foresight dynamics of inflation and output gap, so that the $\{\epsilon_{At}\}$ and $\{\epsilon_{Gt}\}$ realizations that underlie the natural-rate's trajectory are in period-t information sets.

First-Order Conditions for Optimal Policy We now characterize the nonlinear optimal policy problem. Before going further, note that the first-order conditions for R_t can be rearranged to yield

$$\omega_{7t} = \omega_{6t} C_t^{-\sigma^{-1}}$$

using the complementarity condition of $\omega_{7t} = \omega_{7t} R_t$. Furthermore, note that $R_t = 1$ and $\omega_{7t} > 0$ under ZLB and $R_t > 1$ and $\omega_{7t} = 0$ in the absence of ZLB. We thus set $R_t = 1$ and ω_{6t}

Table 3: *An augmented Lagrangian for Optimal Policy*

$$L(\Delta_{-1}, \tilde{\omega}_{-1}, \{s_t\}_{t=0}^{\infty}) =$$

$$\min_{\{\omega_t\}_{t=0}^{\infty}} \max_{\{d_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ [(C_t^{1-\sigma^{-1}} - 1)/(1-\sigma^{-1}) \text{ - } \chi_0 H_t^{1+\chi}/(1+\chi)$$

$$+ \ \omega_{1t} \ (A_t H_t/\Delta_t \text{ - } C_t \text{ - } G_t)$$

$$\text{- } \omega_{2t-1} \ \alpha \Pi_t^{\epsilon-1} Z_{1t} \text{ - } \omega_{2t} \ (A_t H_t C_t^{-\sigma^{-1}}/\Delta_t \text{ - } Z_{1t})$$

$$\text{- } \omega_{3t-1} \ \alpha \Pi_t^{\epsilon} Z_{2t} \text{ - } \omega_{3t}(\ ((\epsilon-1)\chi_0)/\epsilon(1-\tau_t))(H_t^{1+\chi}/\Delta_t) \text{ - } Z_{2t})$$

$$+ \ \omega_{4t} \ ((1-\alpha)(\tfrac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}} + \alpha \Pi_t^{\epsilon} \Delta_{t-1} \text{ - } \Delta_t)$$

$$\text{- } \omega_{5t} \ (Z_{1t} \ (\tfrac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{1}{1-\epsilon}} \text{ - } Z_{2t})$$

$$+ \ \omega_{6t} \ C_t^{-\sigma^{-1}} R_t^{-1} \text{ - } \omega_{6t-1} \ C_t^{-\sigma^{-1}} \ \Pi_t^{-1} + \omega_{7t}(R_t - 1)]$$

Note: In this table, $d_t = \{H_t, \Delta_t, C_t, Z_{1t}, Z_{2t}, \Pi_t, R_t\}$ is a vector of decision variables at period t. In addition, $\omega_t = \{\omega_{1t}, \omega_{2t}, \omega_{3t}, \omega_{4t}, \omega_{5t}, \omega_{6t}, \omega_{7t}\}$ is a vector of Lagrange multipliers chosen at period t. The optimal policy problem in this table solves the Lagrangian given nonstochastic paths for exogenous government consumption, subsidy rate, and aggregate productivity $\{s_t\}_{t=0}^{\infty} = \{G_t, A_t, \tau_t\}_{t=0}^{\infty}$, a set of initial values of Lagrange multipliers $\tilde{\omega}_{-1} = \{\omega_{2-1}, \omega_{3-1}, \omega_{6-1}\}$, and an initial value of relative price distortion Δ_{-1}. The steady-state value of τ_t is set to be one that extinguishes the static monopolistic distortion, so that the nonstochastic steady state with zero inflation can attain the efficient allocation.

> 0 under ZLB and $\omega_{6t} = 0$ in the absence of ZLB based on the positivity of the household's consumption.

The first-order conditions with respect to consumption, hours worked, relative price distortion, gross goods inflation, and the variables Z_{1t} and Z_{2t}, are

$$C_t^{-\sigma^{-1}} - \omega_{1t} + \sigma^{-1}(\frac{A_t H_t}{\Delta_t})C_t^{-(1+\sigma^{-1})}\omega_{2t} + \sigma^{-1}\omega_{6t-1}C_t^{-(\sigma^{-1}+1)}\Pi_t^{-1} - \sigma^{-1}R_t^{-1}\omega_{6t}C_t^{-(\sigma^{-1}+1)} = 0$$

$$-\chi_0 H_t^{\chi} + \omega_{1t}\frac{A_t}{\Delta_t} - \omega_{2t}\frac{A_t}{\Delta_t C_t^{\sigma^{-1}}} - \omega_{3t}\frac{\chi_0(1+\chi)H_t^{\chi}}{(1-\tau_t)(1-\epsilon^{-1})\Delta_t} = 0$$

$$\frac{A_t H_t \omega_{2t}}{\Delta_t^2 C_t^{\sigma^{-1}}} + \frac{\chi_0 H_t^{1+\chi}\omega_{3t}}{(1-\tau_t)\left(1-\epsilon^{-1}\right)\Delta_t^2} - \omega_{4t} + \alpha\beta\omega_{4t+1}\Pi_{t+1}^{\epsilon} = \frac{A_t H_t \omega_{1t}}{\Delta_t^2}$$

$$\begin{aligned}
&-\alpha[(\epsilon-1)\omega_{2t-1}\Pi_t^{\epsilon-2}Z_{1t} + \epsilon\omega_{3t-1}\Pi_t^{\epsilon-1}Z_{2t}] \\
&-\alpha\epsilon\left(\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{\epsilon-1}} - \Pi_t\Delta_{t-1}\right)\omega_{4t}\Pi_t^{\epsilon-2} - \frac{\alpha\Pi_t^{\epsilon-2}}{1-\alpha}\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{-\epsilon}{\epsilon-1}} Z_{1t}\omega_{5t} \\
&+\omega_{6t-1}C_t^{-\sigma^{-1}}\Pi_t^{-2} = 0
\end{aligned}$$

$$\omega_{2t} - \alpha\omega_{2t-1}\Pi_t^{\epsilon-1} - \omega_{5t}(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{1}{1-\epsilon}} = 0 \tag{26}$$

$$\omega_{3t} - \alpha\omega_{3t-1}\Pi_t^{\epsilon} + \omega_{5t} = 0 \tag{27}$$

Loglinear Approximations to the First-Order Conditions We now make a combined loglinear/linear approximation to the Ramsey problem. We first note that several Lagrange multipliers take zero value in the nonstochastic steady state with zero inflation: $\omega_2 = \omega_3 = \omega_5 = \omega_6 = 0$. Notice that this implies that the markup shock, τ_t does not appear in the linearized version of the nonlinear first-order conditions. But because we want the approximation to the model to be faithful to the fact that these series do fluctuate around their mean, we take *linear approximations* for those expressions where the multipliers enter; and use loglinear approximations to the remaining variables in the model. Let $\hat{x}_t$ refer to log-deviations of the variable X_t from its steady-state level $\overline{X}$. We assume that the optimal monetary policy conditions on fiscal decisions, in the sense that the Ramsey policymaker takes government purchases as preordained, not as a choice variable. The conditions that appear in the approximated model are:

$$-\sigma^{-1}\hat{C}_t - \hat{\omega}_{1t} + \frac{\sigma^{-1}}{s_c}\omega_{2t} - \sigma^{-1}\beta C^{-(\sigma^{-1}+1)}(\omega_{6t} - \beta^{-1}\omega_{6t-1}) = 0 \quad (28)$$

$$-\chi\hat{H}_t + \hat{\omega}_{1t} + \hat{A}_t - \omega_{2t} - (1+\chi)\omega_{3t} = 0 \quad (29)$$

$$\frac{C^{1-\sigma^{-1}}}{\omega_4 s_c}\omega_{2t} - \frac{\chi_0 H^{1+\chi}}{(1-\tau)(1-\epsilon^{-1})\omega_4}\omega_{3t} - \hat{\omega}_{4t} + \alpha\beta E_t[\hat{\omega}_{4t+1} + \epsilon\hat{\Pi}_{t+1}] = \hat{A}_t + \hat{H}_t + \hat{\omega}_{1t} \quad (30)$$

$$\hat{\Pi}_t = -(\frac{1-\alpha}{\alpha\epsilon\omega_4})\omega_{6t-1} + \frac{1}{\epsilon}(\omega_{2t} - \omega_{2t-1}) \quad (31)$$

$$\omega_{2t} = \alpha\omega_{2t-1} + \omega_{5t} \quad (32)$$

$$\omega_{3t} = \alpha\omega_{3t-1} - \omega_{5t}, \quad (33)$$

s_c being the steady-state share of government purchases in total spending.

It can be established that the optimal inflation rate is unaffected by the presence of the productivity and public spending shocks. Proceeding toward this goal, let us amalgamate the first two conditions given above:

$$\sigma^{-1}\hat{C}_t + \chi\hat{H}_t - \hat{A}_t + (1 - \frac{\sigma^{-1}}{s_c})\omega_{2t} + (1+\chi)\omega_{3t} + \sigma^{-1}\beta C^{-(\sigma^{-1}+1)}(\omega_{6t} - \beta^{-1}\omega_{6t-1}) = 0 \quad (34)$$

and write out explicitly the output gap identity, corresponding to logarithmic difference between equilibrium GDP under sticky prices with the (log-deviation for the) flexible-price output level:

$$x_t = \hat{Y}_t - \hat{Y}_t^* \quad (35)$$

We also have a number of "gap" relations among the model's loglinearized variables. The resource constraint and technology reveal a proportionality between output, consumption, and hours gaps:

$$x_t = s_c(\hat{C}_t - \hat{C}_t^*) = \hat{H}_t - \hat{H}_t^* \quad (36)$$

Labor hiring (implying equality of marginal product of labor and marginal rate of substitution) and the resource constraint imply that under flexible prices, we have

$$\sigma^{-1}\hat{C}_t^* + \chi\hat{H}_t^* = \hat{A}_t \quad (37)$$

In addition, loglinear expressions for flexible-price consumption and output are:

$$\hat{C}_t^* = -\frac{\chi}{(\sigma^{-1}/s_c)+\chi}\hat{g}_t + \frac{\chi+1}{(\sigma^{-1}/s_c)+\chi}\hat{A}_t; \quad \hat{Y}_t^* = \frac{(\sigma^{-1}/s_c)}{(\sigma^{-1}/s_c)+\chi}\hat{g}_t + \frac{\chi+1}{(\sigma/s_c)+\chi}\hat{A}_t \tag{38}$$

Deploying these relations, we can rewrite the first-order condition for output as:

$$((\sigma^{-1}/s_c)+\chi)x_t + (1-(\sigma^{-1}/s_c))\omega_{2t} + (1+\chi)\omega_{3t} + \sigma^{-1}\beta C^{-\sigma^{-1}-1}(\omega_{6t} - \beta^{-1}\omega_{6t-1}) = 0. \tag{39}$$

Optimal conditions for ω_{2t} and ω_{3t} further imply:

$$\omega_{2t} + \omega_{3t} = \alpha(\omega_{2t-1} + \omega_{3t-1}) \tag{40}$$

It is convenient to define $\omega_{8t} = \omega_{2t} + \omega_{3t}$ and its definition implies the law of motion,

$$\omega_{8t} = \alpha\omega_{8t-1.} \tag{41}$$

Therefore, the Ramsey policy that starts with period 0 conditions entails $\omega_{8t} = 0$, $t = 0, \cdots, \infty$. Taking this relation into account, and rearranging the preceding optimality conditions, we obtain:

$$x_t - \omega_{2t} + \frac{\sigma^{-1}\beta C^{-\sigma^{-1}-1}}{(\sigma^{-1}/s_c)+\chi}(\omega_{6t} - \beta^{-1}\omega_{6t-1}) = 0. \tag{42}$$

A relation for inflation is implied among the conditions above:

$$\pi_t = (\frac{1-\alpha}{\alpha\epsilon\omega_4})\omega_{6t-1} - \frac{1}{\epsilon}(\omega_{2t} - \omega_{2t-1}) \tag{43}$$

where π_t $(= \hat{\Pi}_t)$ is the log-change in the aggregate goods price level between period t and $t-1$. The productivity and fiscal shocks are absent from this pair of equations, which fully capture the policymaker's optimality conditions.

The policy rules for the output gap and inflation implied by Ramsey policy thus apply irrespective of the specification of the productivity and government purchase shocks, and thus make no direct reference to these shocks.

Having characterized the first-order approximation to optimal monetary policy, we now demonstrate that this representation is equivalent to the optimal policy problem using a linear-quadratic approach.

Equivalence of Loglinearized (Nonlinear) Ramsey Solution and Linear-Quadratic Approaches To characterize optimal policy in a linear-quadratic version of the problem, we first take note of the fact that the two constraints for optimal policy are the New Keynesian Phillips curve and the intertemporal IS equation:

$$\pi_t = \beta\ E_t\{\pi_{t+1}\} + \kappa x_t \tag{44}$$

$$x_t = E_t\{x_{t+1}\} - \sigma E_t\{i_t - \pi_{t+1}\} \tag{45}$$

Following Rotemberg and Woodford (1997) and Woodford (2003), an approximation of household utility can be written as:

$$\sum_{t=0}^{\infty} \beta^t E_0[\frac{\pi_t^2}{2} + \lambda \frac{x_t^2}{2}], \tag{46}$$

where we have normalized the weight on inflation variability, leaving $\lambda = \kappa/\epsilon$ as the weight on inflation variability.

Let us define

$$\phi_{1t} = \frac{\lambda \sigma^{-1} \beta C^{-\sigma^{-1}-1}}{(\sigma^{-1}/s_c) + \chi} \omega_{6t}; \quad \phi_{2t} = \frac{\lambda}{\kappa} \omega_{2t} \tag{47}$$

where $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha((\sigma^{-1}/s_c)+\chi)}$. Substituting these expressions into the IS and Phillips curves, we obtain:

$$\pi_t = (\frac{s_c \sigma}{\beta}) \phi_{1t-1} - (\phi_{2t} - \phi_{2t-1}) \tag{48}$$

$$\lambda x_t = \kappa \phi_{2t} - (\phi_{1t} - \beta^{-1} \phi_{1t-1}). \tag{49}$$

These correspond to the first-order conditions associated with the linear-quadratic problem outlined above, and ϕ_{1t} and ϕ_{1t} denote the Lagrange multipliers attached to equations (44) and (45), respectively.

Implementation of the Equivalence Result under Perfect Foresight: Piecewise Linear Approximation Let us consider the solution procedure for the case where, in period zero, a negative shock to the natural real rate of interest strikes. In order to obtain a nonlinear solution of the Ramsey problem under these circumstances, we begin with a guess on the date at which optimal policy is able to launch from the ZLB. Denote this conjectured date T_N^*. For given T_N^*, we partition history into a ZLB phase and post-ZLB phase. Having solved for post-ZLB dynamics, we move on to ZLB-era dynamics. We compute Lagrange multiplier values in order to verify that the value of the Lagrange multiplier attached to the IS equation is in line with our conjectured T_N^* value. The linear/loglinear approximation is used in our analysis once the conjecture matches the actual ZLB departure date.