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Dynamic Allocation Decisions in the Presence of Liability Constraints

Preliminary Version – Do not Quote

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Abstract: This paper introduces a continuous-time model of intertemporal asset-liability management. Using the value of the liability portfolio as a numeraire, we provide explicit solutions in the CRRA case, both in a complete market setting and in an incomplete market setting where liability risk is not entirely spanned by existing securities. When constraints on the funding ratio are introduced, the optimal policy is shown to involve a particular kind of dynamic trading strategy which is reminiscent of Constant Proportion Portfolio Insurance (CPPI) strategies, extended to a relative risk context, where the value of liability is used as a benchmark. We also provide empirical evidence that such dynamic trading strategies can add significant value in terms of risk management for pension plans. These insights have important potential implications in the context of the current pension fund crisis.

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1. Introduction

Recent difficulties have drawn attention to the risk management practices of institutional investors in general and defined benefit pension plans in particular. A perfect storm of adverse market conditions around the turn of the millennium has devastated many corporate defined benefit pension plans. Negative equity market returns have eroded plan assets at the same time as declining interest rates have increased market-to-market value of benefit obligations and contributions. In extreme cases, this has left corporate pension plans with funding gaps as large as or larger than the market capitalization of the plan sponsor. For example, in 2003, the companies included in the S&P 500 and the FTSE 100 index faced a cumulative deficit of $225 billion and £55 billion, respectively (Credit Suisse First Boston (2003) and Standard Life Investments (2003)), while the worldwide deficit reached an estimated 1,500 to 2,000 billion USD (Watson Wyatt (2003)). For some companies, pension deficit is (much) larger than market cap, a well-known example being the United Airlines with a pension fund deficit amounting to $ 9.8 billion by mid 2005 with a market cap well under one billion.[1]

That institutional investors in general, and pension funds in particular, have been so dramatically affected by recent market downturns can be taken as an indication that asset allocation strategies implemented in practice may not be consistent with a sound liability risk management process. In particular, it has often been argued that the kinds of asset allocation strategies implemented in practice, which used to be heavily skewed towards equities in the absence of any protection with respect to their downside risk, were not consistent with a sound liability risk management process. According to an annual survey conducted by LCP (Lane, Clark & Peacock Actuaries & Consultants), it turns out that by 1992, % holdings in equities by pension funds were 75% in the UK, 47% in the US, 18% in the Netherlands and 13% in Switzerland. In 2001, midway through the bear market, pension funds had 64% of their total assets in equities in the UK, 60% in the US, 50% in the Netherlands, and 39% in Switzerland. As a result of such a domination of equities, the increase in liability value that followed decrease in interest rates was only partially offset by the parallel increase in the value of the bond portfolio.

One question that naturally arises is whether the crisis could have been avoided by better asset allocation decisions. Another related question is whether it is possible to improve in the current situation and help solve the pension fund crisis by sound asset allocation practices. Academic research on asset allocation in the presence of liability constraints (also known as asset-liability management) has focused on extending Merton's intertemporal selection analysis (see Merton (1969, 1971)) to account for the presence of

[1] Declared by PBGC on 22 Jun. 2005, on a hearing held by the U.S. House Subcommittee on Aviation.

liability constraints in the asset allocation policy. A first step in the application of optimal portfolio selection theory to the problem of pension funds has been taken by Merton (1990) himself, who studies the allocation decision of a University that manages an endowment fund. In a similar spirit, Rudolf and Ziemba (2004) have formulated a continuous-time dynamic programming model of pension fund management in the presence of a time-varying opportunity set, where state variables are interpreted as currency rates that affect the value of the pension's asset portfolio. Also related is a paper by Sundaresan and Zapatero (1997), which is specifically aimed at asset allocation and retirement decisions in the case of a pension fund. Also related is a recent paper by Binsbergen and Brandt (1995) who study an optimal allocation problem in the presence of liability constraints in a discrete-time setting, with an emphasis on how various regulations with respect to the valuation of liabilities impacts investment decisions.

In a nutshell, the main insight from this strand of the literature is that the presence of liability risk induces the introduction of a specific hedging demand component in the optimal allocation strategy, as typical in intertemporal allocation decisions in the presence of stochastic state variables. On the other hand, one key ingredient that is somewhat missing in the existing literature is how would the presence of hard constraints on the funding ratio (loosely defined as the ratio of the market value of assets over some liability) affect the optimal strategy.

The introduction of funding ratio constraints is not only an obviously desirable feature from a risk management standpoint, but has also been the focus of recent regulation in most developed countries. For example, in the United States, the Pension Benefit Guaranty Corporation (PBGC), which provides a partial insurance of pensions, charges a higher premium to funds reporting a funding level of under 90% of current liabilities, thus providing strong incentives for maintaining the funding ratio over that minimum 90% threshold. In the UK, there was a formal general Minimum Funding Requirement (MFR) that came into effect in 1995, which eventually was replaced in the 2004 Pensions Bill with a scheme-specific statutory funding objective to be determined by the sponsoring firm and fund trustees. A regulatory requirement over a minimum funding ratio rule is also present in other European countries, e.g., in Germany where Pensionskassen and Pensionsfonds must be fully funded at all times to the extent of the guarantees they have given, in Switzerland where the minimum funding level is 100%, with an incentive to conservative management (investment in equities, for example, is limited to 30% of total assets for funds with less than 110% coverage ratio), or in the Netherlands where the minimum funding level is 105% plus additional buffers for investment risks.

This paper introduces a continuous-model for intertemporal allocation decisions in the presence of liabilities. We cast the problem in an incomplete market setting where liability risk is not spanned by existing securities so as to account for the presence of non-hedgeable (e.g., actuarial) sources of risk in liability streams. Using the martingale approach, where we treat the liability portfolio as a natural numeraire, we provide explicit solutions in the unconstrained, as well as when explicit or implicit funding ratio constraints are imposed. In the unconstrained case, we confirm that the optimal strategy involves a fund separation theorem that legitimates investment in the standard efficient portfolio and a liability hedging portfolio, which is consistent with existing research on the subject, and also rationalizes some so-called *liability-driven investment* (LDI) solutions recently launched by several investment banks and asset management firms. Our main contribution is to show that the constrained solution, on the other hand, involves a *dynamic*, as opposed to *static*, allocation to the standard efficient portfolio and a liability hedging portfolio. These strategies are reminiscent of CPPI portfolio insurance strategies which they extend to a relative (with respect to liabilities) risk context. From a technical standpoint, one additional contribution of this paper is to present a new example of the usefulness of change of numeraire techniques, heavily used in asset pricing problems, in the context of a portfolio allocation problem.

Because of its focus on asset allocation decisions with a liability benchmark, our paper is also strongly related to dynamic asset allocation models with performance benchmarks. Single-agent portfolio allocation models with benchmark constraints include notably Browne (2000) in a complete market setting, or Tepla (2001) who also includes constraints on relative performance. Another formally related paper is Brennan and Xia (2002) who study in an incomplete market setting asset allocation decisions when an inflation index is used as a numeraire. Equilibrium implication of the presence of performance benchmarks are discussed in Cuoco and Kaniel (2003), Gomez and Zapatero (2003), or Basak, Shapiro and Tepla (2002). Our paper is also related to the literature on portfolio decisions with minimum target terminal wealth, including Grossman and Vila (1989), Cox and Huang (1989), Basak (1995), or Grossman and Zhou (1996).

The rest of the paper is organized as follow. In section 2, we introduce a formal continuous-time model of asset-liability management. In section 3, we solve the problem in an incomplete market setting and in the absence of explicit or implicit constraints on the funding ratio. In section 4, we introduce such constraints and derive the optimal allocation strategy in a complete market setting. We also present a series of numerical illustrations. In section 5, we present a conclusion as well as suggestions for further research, while technical detailed and proofs of the main results are relegated to as dedicated appendix.

2. A Continuous-Time Model of Asset-Liability Management

In this section, we introduce a general model for the economy in the presence of liability constraints. Let *[0,T]* denote the (finite) time span of the economy, where uncertainty is described through a standard probability space *(Ω,A,P)* and endowed with a filtration $\{F_t; t \geq 0\}$, where $F_\infty \subset A$ and F_0 is trivial, representing the *P*-augmentation of the filtration generated by the *n*-dimensional Brownian motion $(W^1,...,W^n)$.

2.1. Stochastic Model for Asset Prices

We consider *n* risky assets, the prices of which are given by :

$$dP_t^i = P_t^i\left(\mu_i dt + \sum_{j=1}^{n}\sigma_{ij} dW_t^j\right), i = 1,...,n$$

We shall sometimes use the shorthand vector notation for the expected return (column) vector $\mu = (\mu_i)'_{i=1,....,n}$ and matrix notation $\sigma = (\sigma_{ij})_{i,j=1,....,n}$ for the asset return variance-covariance matrix. We also denote **1**=(1,…,1)' a n-dimensional vector of ones and by $W = (W^j)'_{j=1,....,n}$ and the vector of Brownian motions. A risk-free asset, the *0th* asset, is also traded in the economy. The return on that asset, typically a default free bond, is given by $dP_t^0 = P_t^0 rdt$, where *r* is the risk-free rate in the economy.

We assume that *r,* μ and σ are progressively-measurable and uniformly bounded processes, and that σ is a non singular matrix that is also progressively-measurable and bounded uniformly.[2] For some numerical applications below, we will sometimes treat these parameter values as constant.

Under these assumptions, the market is complete and arbitrage-free and there exists a unique equivalent martingale measure ***Q***. In particular, if we define the *risk premium* process $\theta_t = \sigma_t^{-1}(\mu_t - r_t\mathbf{1})$, then we

[2] More generally, one can make expected return and volatilities of the risky assets, as well as the risk-free rate, depend upon a multi-dimensional state variable *X*. These states variables can be thought of various sources of uncertainty impacting the value of assets and liabilities. In particular, one may consider the impact of stochastic interest rate on the optimal policy.

have that the process $Z(0,t) = \exp\left(-\int_0^t \theta_s^{'} dW_s - \frac{1}{2}\int_0^t \theta_s^{'}\theta_s ds\right)$ is a martingale, and ***Q*** is the measure with a Radon-Nikodym density $Z(0,t)$ with respect to the historical probability measure ***P*** (see for example Karatzas (1996)).

By Girsanov theorem, we know that the n-dimensional process defined by $\left(W_t^Q\right)_{t\geq 0} = \left(W_t + \int_0^t \theta_s ds\right)_{t\geq 0}$ is a martingale under the probability Q.[3] The dynamics of the price process can thus be written as:

$$\frac{dP_t}{P_t} = rdt + \sigma dW_t^Q = rdt + \sigma\left(dW_t + \theta dt\right) \qquad (1)$$

2.2. Stochastic Model for Liabilities

We also introduce a separate process that represents in a reduced-form manner the dynamics of the present value of the liabilities:

$$dL_t = L_t\left(\mu_L dt + \sum_{j=1}^{n} \sigma_{L,j} dW_t^j + \sigma_{L,\varepsilon} dW_t^{\varepsilon}\right)$$

where $\left(W_t^{\varepsilon}\right)$ is a standard Brownian motion, uncorrelated with W, that can be regarded as the projection residual of liability risk onto asset price risk and represents the source of uncertainty that is specific to liability risk, emanating from various factors such as uncertainty in the growth of work force, uncertainty in mortality and retirement rates, etc.

The integration of the above stochastic differential equation gives $L_T = L_t \eta(t,T)\eta_L(t,T)$, with:

$$\eta(t,T) \equiv \exp\left\{\int_t^T \left(\mu_L(s) - \frac{1}{2}\sigma_L^{'}(s)\sigma_L(s)\right)ds + \int_t^T \sigma_L^{'}(s) dW_s\right\}$$

$$\eta_L(t,T) \equiv \exp\left\{-\int_t^T \frac{1}{2}\sigma_{L,\varepsilon}^2(s) ds + \int_t^T \sigma_{L,\varepsilon}(s) dW_s^{\varepsilon}\right\}$$

[3] This is provided that the Novikov condition $E_0\left[\exp\left\{\frac{1}{2}\int_0^T \theta_t^{'}\theta_t dt\right\}\right] < \infty$ holds, which is the case for example when all parameter values are bounded functions of t, and of course as a trivial specific case when all parameter values are constant.

When $\sigma_{L,\varepsilon} = 0$, then we are in a complete market situation where all liability uncertainty is spanned by existing securities. Because of the presence of non-financial (in particular actuarial) sources of risk, such a situation is not to be easily expected in practice, and the correlation between changes in value in the liability portfolio and the liability-hedging portfolio (i.e., the portfolio with the highest correlation with liability values) is always strictly lower than one for all pension plans.[4]

if such a hypothetical perfect liability-hedging asset is present in the market place, and assuming for example it is the n^{th} asset, we then have $\mu_L = \mu_n$, $\sigma_{L,j} = \sigma_{n,j}$ for all *j=1, ..., n*, and $\sigma_{L,\varepsilon} = 0$. In general however, $\sigma_{L,\varepsilon} = 0$ and the presence of liability risk that is not spanned by asset prices induces a specific form of market incompleteness. The following proposition provides an explicit characterization of the set of equivalent martingale measures in the presence on non-spanned liability risk.

Proposition 1

The set of all measures under which discounted prices are martingales, where the risk-free asset is used a numeraire, is given by:

$$A = \left\{ Q;\ \exists\ \theta_L\ s.t.\ \frac{dQ}{dP}(t) = Z(0,t) \times Z_L(0,t) \right\}$$

with:

$$Z(0,t) = \exp\left(-\int_0^t \theta_s^{'} dW_s - \frac{1}{2}\int_0^t \theta_s^{'} \theta_s ds \right)$$

$$Z_L(0,t) = \exp\left(-\int_0^t \theta_L(s) dW_s^{\varepsilon} - \frac{1}{2}\int_0^t \theta_L^2(s) ds \right)$$

As $\left(W_t^{\varepsilon}\right)$ is a standard Brownian motion uncorrelated to *W*, a convenient multiplicative separation of asset price and liability risk-adjustments exists. On the one hand, the market price for risk process $\left(\theta_t\right)_{t \geq 0}$ solely affects the asset return process and has no impact on the reward for pure liability risk. On the other hand,

[4] It should be noted that investment banks have recently started to issue customized OTC derivatives (interest rate and inflation swaps for the hedging of financial risk, as well as *mortality bonds* for the hedging of actuarial risks) which are aimed at improving the match between assets and liabilities.

the process $(\theta_L(t))_{t\geq 0}$, which can be interpreted as the *market price for liability risk*, does not affect the return process. The absence of pure liability risk implies that there exists a replicating portfolio w_L such that $L_t = w_L^{'} P_t$ and we then have $\frac{dL_t}{L_t} = rdt + w_L^{'}\sigma\left(dW_t^Q\right) = rdt + w_L^{'}\sigma\left(dW_t + \theta dt\right)$ with $w_L^{'}\sigma = \sigma_L^{'}$.

When $\sigma_{L,\varepsilon} \neq 0$, we have that:

$$\frac{dL_t}{L_t} = rdt + \sigma_L^{'} dW_t^{Q_L} + \sigma_{L,\varepsilon} dW_t^{Q_L} = rdt + \sigma_L^{'}\left(dW_t + \theta dt\right) + \sigma_{L,\varepsilon}\left(dW_t^{\varepsilon} + \theta_L dt\right) \qquad (2)$$

We have that $\mu_L = r + \sigma_L^{'}\theta + \sigma_{L,\varepsilon}\theta_L$, or $\theta_L = \frac{1}{\sigma_{L,\varepsilon}}\left(\mu_L - r - \sigma_L^{'}\theta\right)$. It is only if $\theta_L = 0$ that we have $\mu_L = r + \sigma_L^{'}\theta$ in order to ensure the absence of arbitrage opportunities.

2.3. Objective and Investment Policy

The investment policy is a (column) predictable process vector $\left(w_t^{'} = (w_{1t},\ldots,w_{nt})\right)_{t\geq 0}$ that represents allocations to risky assets, with the reminder invested in the risk-free asset. We define by A_t^w the asset process, i.e., the wealth at time *t* of an investor following the strategy *w* starting with an initial wealth A_0.

We have that: $dA_t^w = A_t^w\left[(1 - w'.\mathbf{1})\frac{dB_t}{B_t} + w'\frac{dP_t}{P_t}\right]$, or: $dA_t^w = A_t^w\left[(r + w'(\mu - r\mathbf{1}))dt + w'\sigma dW_t\right]$.

We now introduce one important state variable in this model, the *funding ratio*, defined as the ratio of assets to liabilities: $F_t = A_t / L_t$.[5] A pension trust has a surplus when the surplus is greater than zero (funding ratio > 100%), fully funded when it is zero (funding ratio = 100%), and under funded when it is less than zero (funding ratio < 100%). In an asset-liability management context, what matters is not the value of the assets per se, but how the asset value compares to the value of liabilities. This is the reason why we suggest using the value of liabilities as a numeraire portfolio in a later section. This is also the

[5] In practice, there exists an ambiguity over whether the liability value is based on using a fixed arbitrary discount rate, or whether a fair value is taken.

reason why it is natural to assume that the (institutional) investor's objective is written in terms of relative wealth (relative to liabilities), as opposed to absolute wealth: $\max_{w} E_0\left[U(F_T)\right]$.

Using Itô's lemma, we can also derive the stochastic process followed by the funding ratio under the assumption of a strategy *w*: $dF_t^w = d\left(\frac{A_t^w}{L_t}\right) = \frac{1}{L_t}dA_t^w - \frac{A_t^w}{L_t^2}dL_t - \frac{1}{L_t^2}dA_t^w dL_t + \frac{A_t^w}{L_t^3}(dL_t)^2$, which yields

$$\frac{dF_t^w}{F_t^w} = \left(\left(r + w'(\mu - r\mathbf{1})\right)dt + w'\sigma dW_t\right) - \left(\mu_L dt + \sigma_L^{'} dW_t + \sigma_{L,\varepsilon} dW_t^{\varepsilon}\right) - \left(w'\sigma\sigma_L dt\right) + \left(\left(\sigma_L^{'}\sigma_L dt + \sigma_{L,\varepsilon}^2\right)dt\right), \text{ or}$$

$$\frac{dF_t^w}{F_t^w} = \left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\varepsilon}^2\right)dt + w'\left(\left(\mu - r\mathbf{1}\right) - \sigma\sigma_L\right)dt + \left(w'\sigma - \sigma_L^{'}\right)dW_t - \sigma_{L,\varepsilon}dW_t^{\varepsilon}.$$

For later use, let us define the following quantities as the mean return and volatility of the funding ratio portfolio, subject to a portfolio strategy *w*:

$$\mu_F^w \equiv \left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\varepsilon}^2\right) + w'\left(\left(\mu - r1\right) - \sigma\sigma_L\right)$$
$$\sigma_F^w \equiv \left(\left(w'\sigma - \sigma_L^{'}\right)'\left(w'\sigma - \sigma_L^{'}\right) + \sigma_{L,\varepsilon}^2\right)^{1/2}$$

3. Solution in the absence of Constraints on the Funding Ratio

In this section, we solve the optimal asset allocation problem in the presence of liability risk using the martingale, or convex duality, approach to portfolio optimization.[6]

3.1. Equivalent Martingale Measures when the Liability Portfolio is used as a Numeraire

Define as $\hat{P}_t^i = P_t^i / L_t$ the time *t* value of the asset *i* when the liability portfolio is used as a numeraire.

<u>*Proposition 2*</u>

[6] It can easily be checked that this solution is identical to the one obtained using the dynamic programming approach.

The set of all probability measures under which prices relative to the liability portfolio (i.e., in an economy where the liability portfolio is used as the numeraire portfolio) are martingales is given by: $A_L = \left\{ Q_L ;\ \exists\ \theta_L \text{ s.t. } \frac{dQ_L}{dP}(t) = \xi(0,t) \times \xi_L(0,t) \right\}$, *with:*

$$\xi(0,t) \equiv \exp\left\{ -\int_0^t \kappa'(s) dW_s - \frac{1}{2}\int_0^t \kappa'(s)\kappa(s) ds \right\}$$

$$\xi_L(0,t) \equiv \exp\left\{ -\int_0^t \kappa_L(s) dW_s^{\varepsilon} - \frac{1}{2}\int_0^t \kappa_L^2(s) ds \right\}$$

Here $\kappa(s) = \theta(s) - \sigma_L(s)\mathbf{1}$, *and* $\kappa_L(s) = \theta_L(s) - \sigma_{L,\varepsilon}(s)\mathbf{1}$ *are the risk premia associated with asset price risk, and pure liability risk, respectively.*

Proof: In the Appendix.

Again, there are an infinite number of equivalent martingale measures unless $\sigma_{L,\varepsilon} = 0$.

3.2. Solution to the Optimization Problem

Solving the optimal allocation problem via a martingale approach involves a two step process. In a first step, one determines the optimal asset value among all possible values that can be financed by some feasible trading strategy. The second step is to determine the portfolio policy financing the optimal terminal wealth. In a complete market setting, the uniqueness of the equivalent martingale measure allows for a simple static budget constraint. In this incomplete market setting, we show that a similar line of reasoning applies, based on the fact that the investor can vary the asset value across states of the world represented by the uncertainty spanned by existing securities. The uncertainty that is specific to liability risk, because it is independent from asset price uncertainty, induces some form of incompleteness that does not directly affect the asset allocation decision.[7]

[7] A formally similar situation can be found in Brennan and Xia (2002) analysis of portfolio selection in the presence of non-spanned inflation risk.

Hence the program reads : $\underset{w_s, t\le s\le T}{Max}\ E_t\left[\dfrac{\left(A_T/L_T\right)^{1-\gamma}}{1-\gamma}\right]$, such that : $E_t^{Q_L}\left[\dfrac{A_T}{L_T}\right]=\dfrac{A_t}{L_t}$, where $\left(A_t\right)$ is financed by a feasible trading strategy with initial investment A_0.

Theorem 1

The optimal terminal funding ratio is given by :

$$F_T^* = \frac{A_T^*}{L_T} = \left(\eta_L(t,T)\right)^{-1}\frac{A_t}{L_t}\left(\xi(t,T)\right)^{-\frac{1}{\gamma}}\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]\right)^{-1}\left(E_t\left[\xi(t,T)\right]^{1-\frac{1}{\gamma}}\right)^{-1}$$

The indirect utility function reads $J_t = \dfrac{\left(F_t\right)^{1-\gamma}}{1-\gamma}g(t,T)$*, with:*

$$g(t,T)=\left(E_t\left[\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}}\right]\right)^{\gamma}\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]\right)^{\gamma-1}E_t\left[\eta_L(t,T)^{\gamma-1}\right]$$

The optimal portfolio strategy is: $w^* = \dfrac{1}{\gamma}(\sigma\sigma')^{-1}(\mu - r\mathbf{1})+\left(1-\dfrac{1}{\gamma}\right)(\sigma')^{-1}\sigma_L$.

Proof: In the Appendix.

We thus obtain a two funds separation theorem, where the optimal portfolio strategy consists of holding two funds, one with weights $w_M = \dfrac{(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}{\mathbf{1}'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}$ and another one with weights $w_L = \dfrac{(\sigma')^{-1}\sigma_L}{\mathbf{1}'(\sigma')^{-1}\sigma_L}$, the rest being invested in the risk-free asset.

The first portfolio is the standard mean-variance efficient portfolio. Note that the amount invested in that portfolio is directly proportional to the investor's Arrow-Pratt coefficient of risk-tolerance $-\dfrac{J_F}{FJ_{FF}}=\dfrac{1}{\gamma}$ (the inverse of the relative risk aversion). This makes sense: the higher the investor's (funding) risk tolerance, the higher the allocation to that portfolio will be.

In order to better understand the nature of the second portfolio, it is useful to remark that it is a portfolio that minimizing the local volatility σ_F^w of the funding ratio. To see this, recall that the expression for the local variance is given by $\sigma_F^w = \left(\left(w'\sigma - \sigma_{L^2}^{'}\right)^{'}\left(w'\sigma - \sigma_L^{'}\right) + \sigma_{L,\varepsilon}^2\right)^{1/2}$, which reaches a minimum for $w^* = (\sigma')^{-1}\sigma_L$, with the minimum being $\sigma_{L,\varepsilon}^2$. As such, it appears as the equivalent of the minimum variance portfolio in a relative return-relative risk space, also the equivalent of the risk-free asset in a complete market situation where liability risk is entirely spanned by existing securities ($\sigma_{L,\varepsilon}^2 = 0$). Alternatively, this portfolio can be shown to have the highest correlation with the liabilities. As such, it can be called a liability-hedging portfolio, in the spirit of Merton (1971) intertemporal hedging demands. Indeed, if we want to maximize the covariance $w'\sigma\sigma_L$ between the asset portfolio and the liability portfolio L, under the constraint that $\sigma_A^2 = w'\sigma\sigma' w$, we obtain the following Lagrangian: $L = w'\sigma\sigma_L - \lambda\left(w'\sigma\sigma' w - \sigma_A^2\right)$. Differentiating with respect to w yields: $\frac{\partial L}{\partial w} = \sigma\sigma_L - 2\lambda\sigma'\sigma w$, with a strictly negative second derivative function. Setting the first derivative equal to zero for the highest covariance portfolio leads to the following portfolio, which is indeed proportional to the liability hedging portfolio $w = \frac{1}{2\lambda}(\sigma\sigma')^{-1}\sigma\sigma_L = \frac{1}{2\lambda}(\sigma')^{-1}\sigma_L$. It should also be noted, as is well-known, that when $\gamma = 1$, i.e., in the case of the log investor, the intertemporal hedging demand is zero (myopic investor).

3.3. Improvement in Investor's Welfare due to Complete versus Incomplete Markets

Note that in the complete market case, $\sigma_{L,\varepsilon} = 0$, and $g(t,T)$ becomes:

$$g_{complete}(T-t) = \exp\left[-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)\kappa'\kappa(T-t)\right]$$

Therefore, the increase in investor's welfare that emanates from completing the market is given by:

$$\frac{g_{complete}(T-t)}{g_{incomplete}(T-t)} = \exp\left[\left(\gamma\left(\sigma_{L,\varepsilon}^2 - \kappa_L\sigma_{L,\varepsilon}\right) - \frac{(1-\gamma)(2-\gamma)}{2}\sigma_{L,\varepsilon}^2\right)(T-t)\right]$$

Here we recall that $\kappa_L = \theta_L = \frac{1}{\sigma_{L,\varepsilon}}\left(\mu_L - r - \sigma_L^{'}\theta\right) = \frac{1}{\sigma_{L,\varepsilon}}(\mu_L - \mu_{LH})$ is the risk premium for pure liability risk, with μ_{LH} is the expected return on the liability-hedging portfolio, and *r* the risk-free rate.

4. Dynamic Portfolio Strategies

4.1. Solution to the Problem with Constraints

We have seen in the previous section that the optimal strategy consists of holding two funds, in addition to the risk-free asset, the standard mean-variance portfolio and the liability hedging portfolio. The proportions invested in these two funds are constant and given by $\frac{\mathbf{1}'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}{\gamma}$ and $\left(1-\frac{1}{\gamma}\right)\mathbf{1}(\sigma')^{-1}\sigma_L$, respectively.

The assumption of a static optimal portfolio strategy involving the standard mean-variance efficient portfolio and the liability-hedging portfolio is only justified under extreme assumptions such as a constant opportunity set and CRRA utility. It would certainly be desirable to introduce stochastic interest rates. This is because the most important source of uncertainty in an ALM problem is interest rate uncertainty. Indeed, time-variation of interest rates have a direct impact on the present value of liabilities; they also have a direct impact on bond prices. One can show that this will induce a demand for a so-called interest rate hedging portfolio. In particular, the above results can actually be extended in a rather straightforward manner to a setup with stochastic Vasicek interest rates. As expected, we then obtain for the optimal strategy a three-fund separation theorem with the introduction of a separate hedging portfolio for interest rate risk.

In what follows, we consider an interesting extension where the optimal strategy involves a dynamic rebalancing of the two afore-mentioned portfolios, even in the absence of a stochastic opportunity set. The ingredient that we introduce is the presence of constraints on the funding ratio, which, as recalled in the introduction, are dominant in pension funds' environment.

To account for the presence of regulatory or otherwise constraints on the funding ratio, one might e tempted to consider the following program :

$$\underset{w_s, t \le s \le T}{Max} E_t \left[\frac{\left(A_T \big/ L_T \right)^{1-\gamma}}{1-\gamma} \right]$$

such that $\frac{A_T}{L_T} \ge k$ almost surely. One problem with such explicit constraints, as argued by Basak (2002) in a different context, is that marginal indirect utility from wealth discontinuously jumps to infinity

So as to provide a smoother taking into account of the presence of constraints on the funding ratio, we instead consider in what follows a program with implicit constraints:

$$\underset{w_s, t \le s \le T}{Max} E_t \left[\frac{\left(\frac{A_T}{L_T} - k \right)^{1-\gamma}}{1-\gamma} \right]$$

Here k again is interpreted as a minimum funding ratio requirement and is assumed to be such that $k \le F_0$ (otherwise, it will not be feasible to ensure the respect of the constraint). In this program, marginal utility goes smoothly to infinity at the minimum funding ratio. Also note that we consider the complete market case in what follows. In the incomplete market setup, the presence of a non hedgeable source of risk will make it impossible for the (implicit) constraint to hold almost surely.

Hence our program reads : $\underset{w_s, t \le s \le T}{Max} E_t \left[\frac{\left(\frac{A_T}{L_T} - k \right)^{1-\gamma}}{1-\gamma} \right]$, such that : $E_t^{Q_L} \left[\frac{A_T}{L_T} \right] = \frac{A_t}{L_t}$, where (A_t) is financed by a feasible trading strategy with initial investment (A_0).

Theorem 2

In the complete market case, the optimal terminal funding ratio is given by :

$$F_T^* = F_t \left[E_t \left(\left(\xi(t,T) \right)^{1-\frac{1}{\gamma}} + k \xi(t,T) \right) \right]^{-1} \left(\xi(t,T) \right)^{-\frac{1}{\gamma}} + k$$

The indirect utility function reads:

$$J_t = \frac{1}{1-\gamma}\left(F_t \left[E_t \left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T) \right) \right]^{-1} \left(\xi(t,T)\right)^{-\frac{1}{\gamma}} + k \right)^{1-\gamma}$$

The optimal portfolio strategy is:

$$w^* = \frac{1}{\gamma}\left(1 - \frac{k}{F_s}\right)(\sigma\sigma')^{-1}(\mu - r\mathbf{1}) + \left(1 - \frac{1}{\gamma}\left(1 - \frac{k}{F_s}\right)\right)(\sigma')^{-1}\sigma_L$$

Proof: In the Appendix.

For a given value of the risk-aversion parameter, it should be note that that the investment in the log-optimal portfolio is lower than in the absence of liability constraints since $\left(1 - \frac{k}{F_t}\right) < 1$ and it is decreasing as the funding ratio decreases towards the threshold level. The recommendation is therefore for pension plans in better financial situation to take more aggressive policies, while plans in worsening situations should undertake less aggressive policies.

It should also be noted that the fraction of wealth allocated to the optimal growth portfolio $w_m = \frac{(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}{\mathbf{1}'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}$ is given by : $\frac{\mathbf{1}'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}{\gamma}\left(A_t - \frac{A_t k}{F_t}\right) = \frac{\mathbf{1}'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}{\gamma}\left(A_t - kL_t\right)$.

Therefore, if we define the *floor* as kL_t, i.e., the value of liability that is consistent with the constraint, and the *cushion* as $A_t - kL$, then we obtain that the investment in the growth optimal portfolio is always equal to a constant multiple *m* of the difference between the asset value and the floor. That constant coefficient is $m = \frac{\mathbf{1}'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}{\gamma}$.

This is strongly reminiscent of CPPI (constant proportion portfolio insurance) strategies, which the present setup extends to a relative risk management context.[8] While CPPI strategies are designed to prevent final terminal wealth to fall below a specific threshold, extended CPPI strategies are designed to protect asset

[8] CPPI strategies have originally been introduced by Black and Jones (1987) and Black and Perold (1992).

value not to fall below a pre-specified fraction of some benchmark value. This result rationalizes and extends the so-called *contingent optimisation* technique, a concept introduced by Leibowitz and Weinberger (1982ab) with no theoretical justification (see also Amenc et al. (2005) for the benefits of dynamic asset allocation strategies in the context of the management of downside risk relative to a benchmark).[9]

On the other hand, it should be noted that the fraction of wealth allocated to the liability hedging portfolio $w_L = \frac{(\sigma')^{-1}\sigma_L}{\mathbf{1}'(\sigma')^{-1}\sigma_L}$ is given by $\frac{(\sigma')^{-1}\sigma_L}{\mathbf{1}(\sigma')^{-1}\sigma_L}\left(A_t - \frac{1}{\gamma}(A_t - kL_t)\right)$. The investment in the liability-hedging portfolio is always equal to a constant multiple *m'* of the difference between the asset value and the cushion divided by the coefficient of risk-aversion. The multiplier coefficient is $m' = \frac{(\sigma')^{-1}\sigma_L}{\mathbf{1}(\sigma')^{-1}\sigma_L}$.

These strategies have a model-free, built-in element of optimality, which is intuitively related to the fact that they allow pension funds to protect their current funding ratio while giving them access to upside potential (and hence the hope for reduction of contributions).

4.2. Numerical Illustrations

In what follows, we present a set of illustrations of the benefits of dynamic allocation strategies in asset-liability management.

Our illustrations are based on a stylized pension fund problem, where the liabilities are assumed to take on a simple form that consists of a series of 20 annual inflation-protected pension payments of equal real value normalized at $100.

To check for the performance of various competing strategies, we use a standard model for generating stochastic scenarios for risk factors affecting asset and liability values; and we generate a set of 1,000 scenarios for interest rates, inflation rate, equity prices as well as real estate prices, when needed. ALM models are typically chosen so as to represent actual behaviors

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[9] The author has collected anecdotic evidence that various firms have started offering services that are similar in spirit to the solution of the optimal portfolio selection problem we consider here. In particular, we refer to State Street Global Advisors (SSgA)'s so-called *Dynamic Risk Allocation Model* (DRAM), as well as AXA-Investment Manager (Axa-IM)'s *Dynamic Contingent Immunization* strategies.

as best as possible and parameters are chosen so as to be consistent with long-term estimates. The next step involves using some optimization technique to find the set of optimal portfolios.

In terms of stochastic scenario simulation, one typically distinguishes between three main risk factors affecting asset and liability values: interest rate risk (or, more accurately, interest rate risks since there are more than one risk factor affecting changes in the shape of the yield curve), inflation risk, and stock price risk. In the illustrations that follow in, we have used a standard model, borrowed from Ahlgrim, D'Arcy and Gorvett (2004), including as key features a two-factor mean-reverting process for real interest rates, a one-factor mean-reverting process for inflation rate, a Markov regime switching model for excess return on equity (excess return). The model is and can be written as:[10]

$$\begin{aligned} dr_t &= a_r(l_t - r_t)dt + \sigma_r dW_t^r \\ dl_t &= a_l(b_l - l_t)dt + \sigma_l dW_t^l \\ d\pi_t &= a_\pi(b_\pi - \pi_t)dt + \sigma_\pi dW_t^\pi \\ dS_t/S_t &= (r_t + \pi_t)dt + b_x^s dt + \sigma_x^s dW_t^x \end{aligned}$$

Here r_t (respectively, π_t) is the real short-term rate (respectively, inflation rate) at date t, a_r (respectively, a_π) the speed of mean reversion of the short-term rate (respectively, inflation rate), l_t (respectively, b_π) is the long-term mean value of the short-term rate (respectively, inflation rate), and σ_r (respectively, σ_π) is the volatility of the short-rate (respectively, inflation rate). This model assumes a particular two-factor process for the real rate so as to account for the non-perfect correlation between bonds of different maturities. In particular, it assumes that the long-term mean value l_t of the short-term rate is also stochastically time-varying, with a speed of mean reversion denoted by a_l, a long-term mean value denoted by b_l) and a volatility denoted by σ_l. By contrast the long-term mean value of the inflation rate is assumed to be a constant. Here W^r, W^l and W^π are three (correlated) standard Brownian motions representing uncertainty driving the three risk-factors. Beside, a Markov-regime switching model is assumed for equity returns, with b_x, the (state-dependent) excess expected return (over the nominal rate $(r_t + \pi_t)$) and σ_x, the (state-dependent) stock volatility. Here W^s is a standard Brownian motion representing uncertainty driving stock returns, and is correlated to W^r, W^l and W^π. The introduction of

[10] Other competing models can of course be used in ALM simulations and optimization, but there are mostly consistent in spirit with this particular model, which we have chosen because it represents a standard example of a state-of-the-art ALM model which is made available for public use by the Casualty Actuarial Society (CAS) and the Society of Actuaries (SOA) (see reference list for exact references of the paper and a web site where the paper can be downloaded).

a Markov regime-switching model is motivated by the desire to fit important empirical characteristics of equity returns, such as the presence of fat-tails and stochastic volatility with volatility clustering effects. The basic idea is that returns are not drawn from a single normal distribution; rather there are two distributions at work generating the returns observed. The equity returns distribution is assumed to jump between two possible states, usually referred to as regimes, denoted by $x=1$ and $x=2$ and interpreted as a low and a high volatility regimes. A transition matrix controls the probability of moving between states.

The parameter values are given in exhibit 1.

Real interest	**Parameter value**
mean reversion speed for short rate process	1
volatility of short rate process	0.01
mean reversion speed for long-term mean value	0.1
volatility of long-term mean value	0.0165
long-term mean reversion level for long-term mean value	0.028
correlation between short-rate and long-term mean value	0.5
Inflation	
mean reversion speed for inflation process	0.4
volatility of inflation process	0.04
long-term mean reversion level for inflation	0.048
correlation between inflation and short-term interest rate	-0.3
Equity model – Regime switching	
(monthly) mean equity excess return in state 1	0.008
(monthly) volatility of equity return in state 1	0.039
(monthly) mean equity excess return in state 2	-0.011
(monthly) volatility of equity return in state 2	0.113
Equity model - Regime switching probabilities	
probability of staying in state 1	0.989
probability of switching from state 1 to state 2	0.011
probability of staying in state 2	0.941
probability of switching from state 2 to state 1	0.059

Exhibit 1: Parameter values – borrowed from Ahlgrim, D'Arcy and Gorvett (2004)

It should be noted that this model is not consistent with the model used in Sections 2 and 3 when deriving the optimal solutions. In particular, because of an attempt to obtain analytical solutions, our base model had a constant interest rate and did not incorporate regime switching patterns in terms of equity returns. The illustrations below actually suggest that simple strategies designed in the context of a very stylized model prove to be rather effective when transposed in a more complex environment.

We take parameter values that are identical to those in Ahlgrim, D'Arcy and Gorvett (2004); who calibrate the model with respect to long time-series. Other choices of parameter values can of course be adopted and their implementation would be straightforward.

4.2.1. Cash-Flow Matching Strategy

One natural solution for meeting the liability constraints consists of buying equal amounts of zero-coupon inflation protected securities (TIPS) with maturities ranging from 1 year to 20 years, assuming they exist (alternatively, OTC interest rate and inflation swap can be used to complement existing cash instruments so as to generate a perfect match with liabilities, here a stream of 20 annual $100 payments). This equally-weighted portfolio of TIPS is the practical implementation of the liability matching portfolio introduced at a conceptual level in section 3.

Using the afore-mentioned stochastic model, and associated parameter values, we generate random paths for the price of 20 zero-coupon TIPS with maturities matching expected payment dates. We find the present value of liability-matching portfolio, denoted as $L(0)$, and we obtain $L(0) = 1777.15$. As we can see the performance is poor, and the burden of contributions is very high: the amount of money needed to generate 20 annual $100 payments is not much smaller than 20x100. This is due to the fact that rates are typically very low. The client needs a very high current contribution to sustain his/her future consumption needs.

On the other hand, one key advantage of this approach, which represents an extreme positioning in the risk-return space, is that the distribution of surplus at date 20 is trivially equal to 0. There is no possible deficit (nor surplus), because the present value of the future liability payments has been invested in a perfect replicating portfolio strategy.

In this context, it is reasonable, unless in the presence of an extremely (infinitely) high risk aversion, to add risky asset classes to enhance the return and decrease the pressure on contributions, at the costs of introducing a risk of mismatch between assets and liabilities. This is what we turn to next.

4.2.2. Surplus Optimization Strategies

We now generate stochastic scenarios also for nominal bonds and stocks. We then start with same initial amount L(0), and find the best fixed-mix strategy that consists of investment in stocks, bonds and liability-matching portfolio (regarded as a whole) so as to generate an efficient frontier in a surplus space based on optimizing the trade-off between expected surplus and variance of the surplus (see the blue line in exhibit 2). Of course, as underlined in section 3, the minimum risk portfolio corresponds to 100% investment in the liability-matching portfolio (corresponding to point A in exhibit 2). Formally, we assume that the asset portfolio is liquidated each year, a liability payment is made, and the remaining wealth is invested in optimal portfolio; in scenarios such that the remaining wealth is not sufficient for making the promised liability payment, we assume that borrowing at the risk-free rate is performed so as to make up for the difference. We estimate probabilities of not meeting the objectives (probability of a deficit), which are reported in exhibit 3, and also plot the distribution of surplus at date 20 for a few points on the efficient frontier (see exhibit 4). As can be seen in exhibit 2, increasing the allocations to stocks and nominal bonds, which have a long-term performance higher than that of inflation-protected bonds but are not as a good a match with respect to liabilities, lead to higher value of the expected surplus, and therefore to average contribution savings, but also to an increased volatility of the surplus and an increased probability of the deficit.

For comparison purposes, we also perform the same exercise and design the efficient frontier when the liability-matching portfolio is not available in the menu of asset classes (see the green line in exhibit 2). The improvement induced by the introduction of a liability-matching portfolio is spectacular, as can be seen by a simple comparison between point A and A’ or B and B’. Regarding point B and B’ for instance, one can see in exhibit 3 that for the same level of expected surplus (€376.78), the volatility of the surplus is increased by more than 50% when the liability-matching portfolio is not available (640.24 versus 423.65). The risk reduction benefits are also spectacular when risk is measured in terms of probability of a deficit or expected shortfall. Intuitively, such a dramatic improvement in investor’s welfare is related to the fact that it is only through the completion of the menu of asset classes that arises from the introduction of a dedicated liability-matching portfolio that the investor’s specific objective and constraints as well as related risk exposures are fully taken into account.

Of course, the difference between optimal portfolios in the presence and in the absence of a liability-matching portfolio is decreasing with the investor's risk-aversion: risk-seeking investors do not seek to enjoy the benefits of liability protection and mostly invest in stocks and bonds anyway.

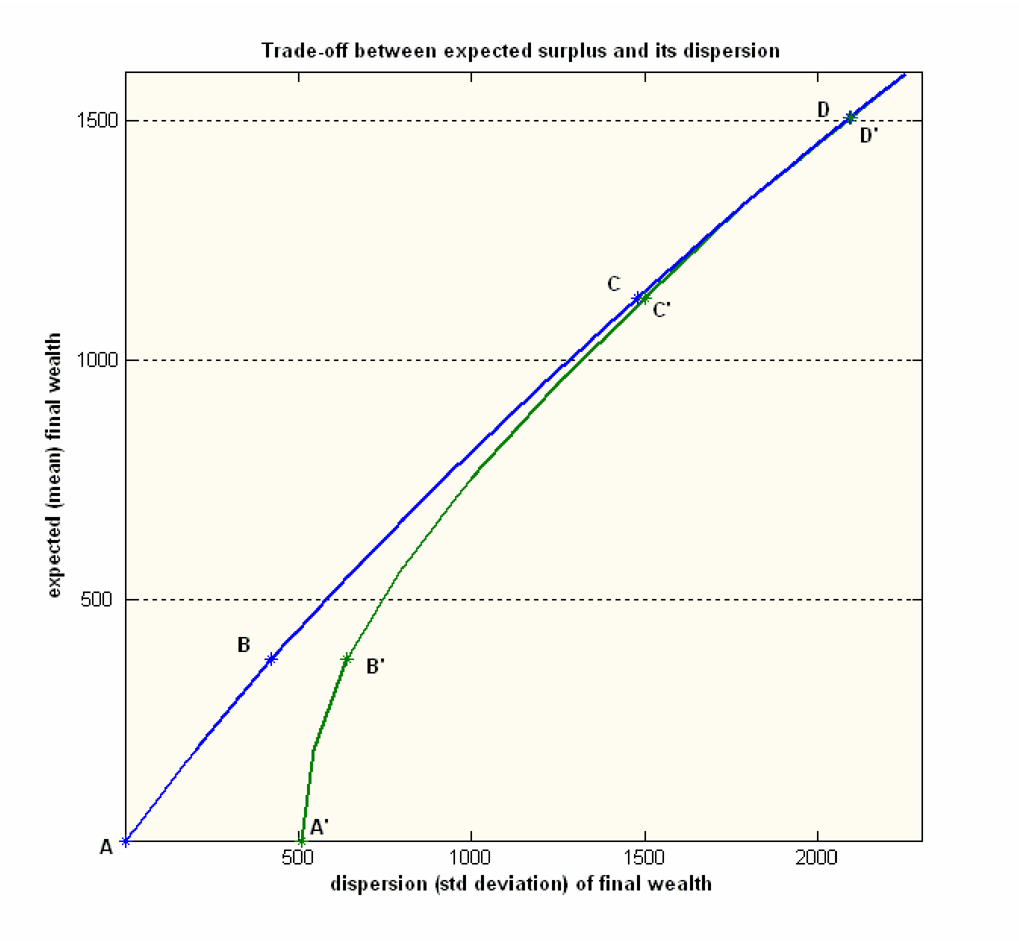

Exhibit 2: Efficient frontier in a mean-variance surplus space

	weights			expected Surplus	volatility of surplus	Prob(S<0)	expected shortfall		necessary nominal contribution	relative Contribution Saving p.a.
	stocks	bonds	Liab-PF							
A	0%	0%	100%	0.00	0.00	0.00%	0.00	(0.00%)	1777.15	0.00%
B	19%	22%	59%	376.78	423.65	15.50%	204.45	(11.50%)	1556.99	12.39%
C	43%	41%	16%	1130.33	1478.91	18.70%	427.92	(24.08%)	1314.07	26.06%
D	52%	45%	3%	1507.11	2093.77	19.60%	502.64	(28.28%)	1233.58	30.59%
A'	10%	90%	0%	0.00	508.92	36.60%	432.34	(24.33%)	1777.15	0.00%
B'	20%	80%	0%	376.78	640.24	25.20%	385.97	(21.72%)	1556.99	12.39%
C'	43%	57%	0%	1130.33	1500.11	19.30%	457.69	(25.75%)	1314.07	26.06%
D'	51%	49%	0%	1507.11	2094.78	20.00%	499.21	(28.09%)	1233.58	30.59%

Exhibit 3: Allocation strategies and risk-return indicators; all values are given as present values at initial date (based on a L(0)=1777.15 initial investment); losses relative to L(0) are reported in parentheses for expected shortfall); the relative contribution saving corresponds to the increase (in percentage) in initial

investment that should have been taken place with a given strategy so as to generate an expected surplus equal to zero.

These ALM optimization exercise consists of finding the portfolios that are optimal from standpoint of protecting the investors' liabilities. A pure asset management (AM) exercise, on the other hand, focuses on designing the portfolios with the optimal risk-return trade-off. Of course, nothing guarantees that AM efficient portfolios will be efficient from an ALM perspective (and vice-versa); in particular, the focus is on nominal return from an AM perspective, while it is on real return from an ALM perspective. To test for the ALM performance of AM efficient portfolios, we have conducted the following experiment. We first find the standard (Markowitz efficient) frontier based on horizon returns, i.e., the portfolios that achieves the lowest level of volatility (across scenarios at horizon) for a given expected return (across scenarios at horizon). We then plot these portfolios (in green) in the (expected surplus-volatility of the surplus) ALM space (see exhibit 5).

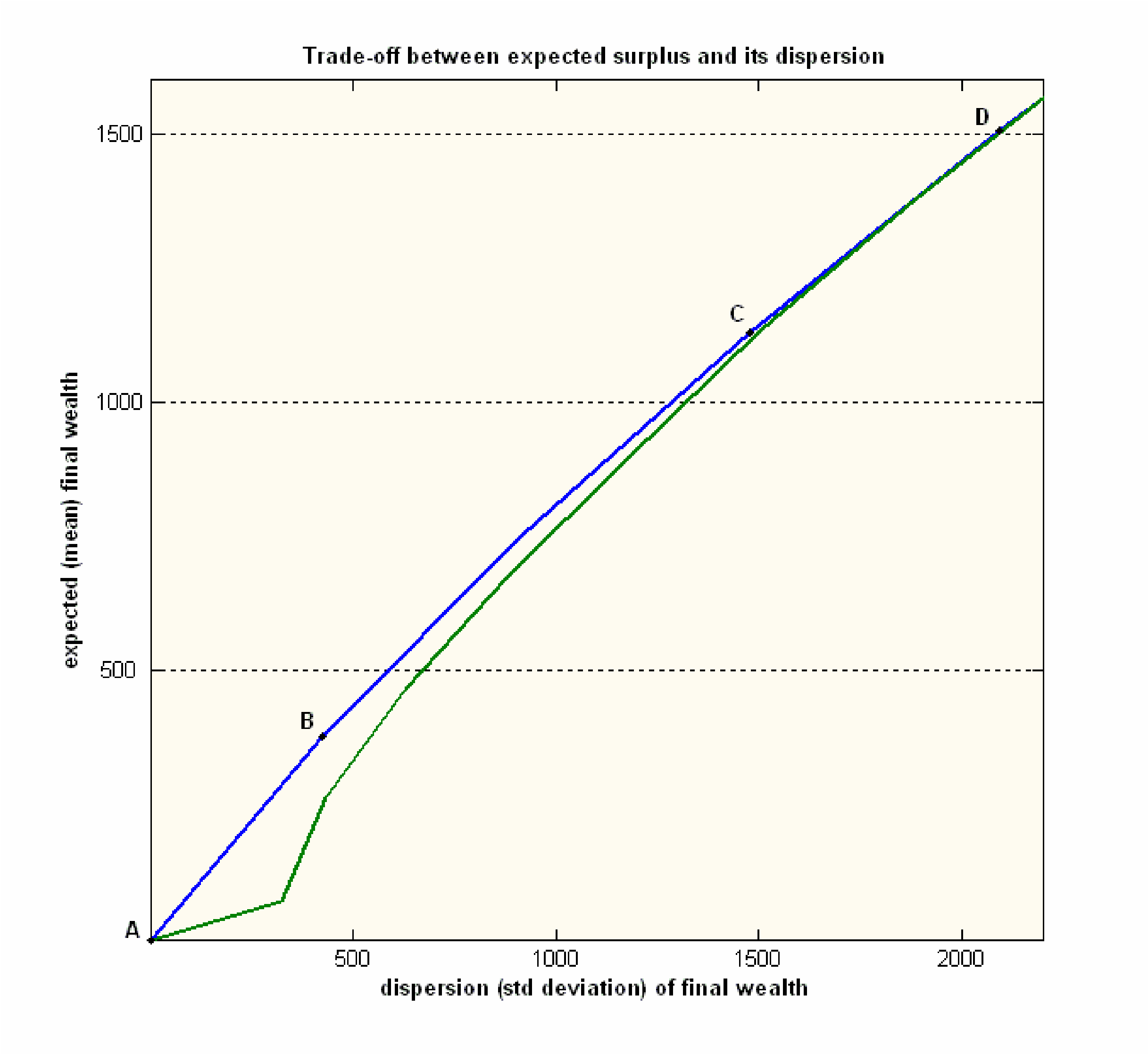

Exhibit 4: AM and ALM efficient frontiers in a mean-variance surplus space

From exhibit 5, we can check that a portfolio efficient in an AM sense is indeed not necessarily efficient in an ALM sense, and vice-versa. Hence, not taking into account liability constraints leads to potentially severe inefficiencies from the investor's standpoint.

We now turn to dynamic portfolio strategies.

4.2.3. Dynamic LDI Strategies

In testing the implementation of the dynamic LDI strategies, the performance portfolio is taken to be the stock-bond portfolio with the highest Sharpe ratio (with our choice of parameter values, and a 4% risk-free rate, we obtain the following portfolio: 28.5% in stocks and 71.5% in bonds), while the liability-matching portfolio is the afore-mentioned portfolio invested in the 20 zero-coupon TIPS with maturities matching expected payment dates.

We consider the extended CPPI strategy introduced in section 4. We consider 6 variants of the strategy, with the level of protection *k=90%,* or *k=95%,* and the multiplier value *m=2, 3* and *4*. The results are reported in exhibits 6 to 9, where we present the performance of the various dynamic strategies and compare them to the performance of their static counterpart. The static counterpart of a given dynamic portfolio strategy is defined as the strategy involving constant (fixed-mix) allocation to the portfolio with highest Sharpe ratio and liability-matching portfolio that matches the initial allocation of the corresponding dynamic strategy. For example, when *k=95%* and *m=4*, the initial allocation to the liability-matching portfolio (respectively the highest Sharpe ratio portfolio) is given by *1-(1-k)m=80%* (respectively, 20%). The static counterpart of the extended CPPI strategy with parameters *k=95%* and *m=4* is therefore a fixed-mix strategy with a constant 80%-20% allocation to liability matching portfolio and performance-seeking portfolio.

dynamic CPPI	expected Surplus	volatility of surplus	Prob(S<0)	expected shortfall	necessary nominal contribution p.a.	relative Contribution Saving p.a.
m=2 k=0.90	121.97	188.42	25.20%	66.45 (3.74%)	1694.19	4.67%
m=3 k=0.90	184.75	326.33	30.20%	97.21 (5.47%)	1658.70	6.66%
m=4 k=0.90	203.97	388.70	36.60%	119.11 (6.70%)	1646.97	7.33%

static CPPI	expected Surplus	volatility of surplus	Prob(S<0)	expected shortfall	necessary nominal contribution p.a.	relative Contribution Saving p.a.
m=2 k=0.90	99.39	110.72	14.90%	80.80 (4.55%)	1706.70	3.96%
m=3 k=0.90	153.12	174.95	15.90%	119.84 (6.74%)	1673.88	5.81%
m=4 k=0.90	209.74	245.63	16.80%	158.93 (8.94%)	1642.37	7.58%

Exhibit 6: risk-return indicators for extended CPPI strategies for a level of guarantee k=90%, as well as for their static counterpart; all values are given as present values at initial date (based on a L(0)=1777.15 initial investment); losses relative to L(0) are reported in parentheses for expected shortfall); the relative contribution saving corresponds to the increase (in percentage) in initial investment that should have been taken place with a given strategy so as to generate an expected surplus equal to zero.

dynamic CPPI	expected Surplus	volatility of surplus	Prob(S<0)	expected shortfall	necessary nominal contribution p.a.	relative Contribution Saving p.a.
m=2 k=0.95	58.48	88.18	25.10%	32.75 (1.84%)	1734.51	2.40%
m=3 k=0.95	94.38	175.82	29.90%	48.12 (2.71%)	1711.43	3.70%
m=4 k=0.95	115.40	240.19	36.80%	58.48 (3.29%)	1698.24	4.44%

static CPPI	expected Surplus	volatility of surplus	Prob(S<0)	expected shortfall	necessary nominal contribution p.a.	relative Contribution Saving p.a.
m=2 k=0.95	48.38	52.58	14.10%	40.43 (2.27%)	1741.04	2.03%
m=3 k=0.95	73.55	80.93	14.30%	61.46 (3.46%)	1723.66	3.01%
m=4 k=0.95	99.39	110.72	14.90%	80.80 (4.55%)	1706.70	3.96%

Exhibit 7: risk-return indicators for extended CPPI strategies for a level of guarantee k=95%, as well as for their static counterpart; all values are given as present values at initial date (based on a L(0)=1777.15 initial investment); losses relative to L(0) are reported in parentheses for expected shortfall); the relative contribution saving corresponds to the increase (in percentage) in initial investment that should have been taken place with a given strategy so as to generate an expected surplus equal to zero.

As can be seen in exhibit 6 and in exhibit 7, most dynamic strategies allow for significantly lower expected shortfall numbers as well as higher expected surplus (and hence higher contribution savings) when compared to their static counterparts. On the other hand, they tend to generate a higher volatility. Also, the probability of a deficit is rather large with dynamic strategies, which aim at avoiding all deficit beyond the minimum threshold (90% or 95%), as opposed to minimizing the probability to face such a relatively low deficit. In essence, dynamic ALM strategies generate asymmetric surplus distributions, as in confirmed by exhibit 8 and 9, where the various surplus distributions are presented. We also note, as expected, that increasing the guaranteed level k and decreasing the multiplier value m lead to more conservative strategies, with less potential for surplus performance and lower risk.

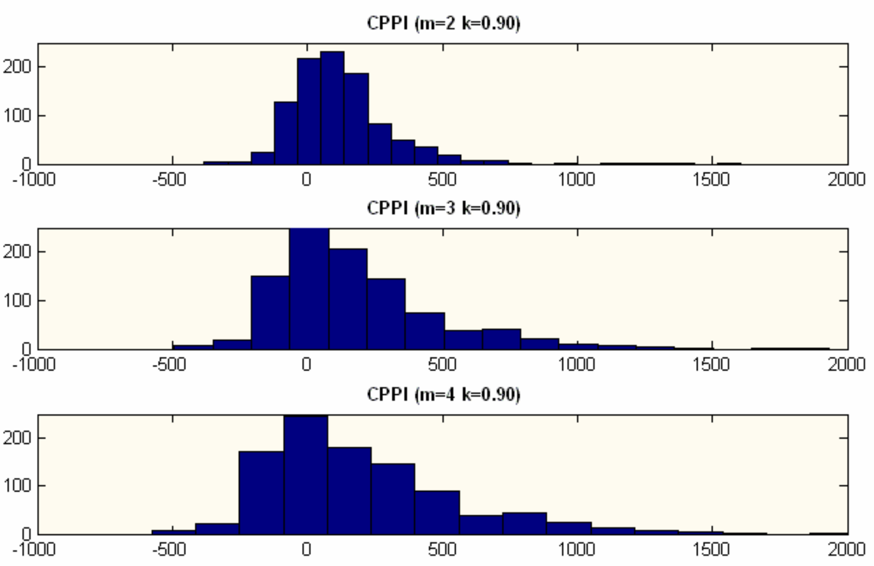

Exhibit 8: Distribution of the final surplus/deficit for extended CPPI strategies for a 90% guarantee level.

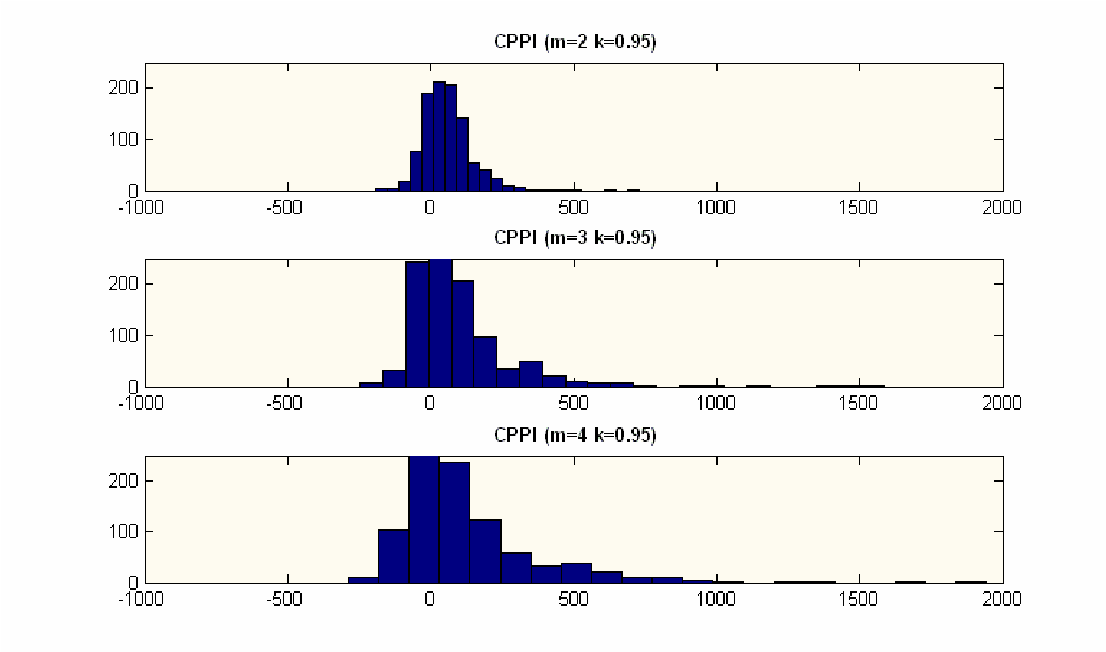

Exhibit 9: Distribution of the final surplus/deficit for extended CPPI strategies for a95% guarantee level.

Overall, the results reported in exhibits 6 to 9 show very significant risk management benefits that arise from dynamic strategies.

5. Conclusion and Extensions

In this paper, we have considered an intertemporal portfolio problem in the presence of liability constraints. Using the value of the liability portfolio as a natural numeraire, we have found that the solution to this problem was cast in terms of non-myopic strategies involving dynamic rebalancing of several funds including in particular.

The contribution of the paper is to show, both from a theoretical standpoint and an empirical standpoint, that a static investment in two "funds", the standard optimal growth portfolio and a liability hedging portfolio, is optimal in the absence of constraints on the funding ratio (fund separation theorem). On the other hand, a dynamic strategy in these two funds, leading to a convex relative payoff reminiscent of portfolio insurance strategies that they extend to a relative risk context, is optimal in the presence of funding ratio constraints. Such extended liability insurance strategies have a model-free, built-in element of optimality, which is the fact that they allow pension funds to protect their current funding ratio (a more sexy proposal indeed when it is higher than 100%) while giving them access to upside potential (and hence the hope for reduction of contributions). In contrast, pure liability hedging strategies, i.e., cash-flow matching or duration matching strategies, which are represented in our framework by a 100% investment in the liability hedging portfolio strategy (consistent with infinite risk aversion), also protect the funding ratio but do not allow for upside potential. On the other hand, investment in a well diversified portfolio of risky assets (consistent with risk aversion equal to 1, the log case) offers access to performance potential but does not protect the funding ratio. Finally, arbitrary fixed-mix strategies (e.g., 70% in bonds and 30% in stocks) such as the ones used in practice appear as downgraded versions of the two-fund portfolio strategies, where the "safe" part of the portfolio (bonds) is not as correlated to liabilities as the pure liability portfolio (unless the liability hedging portfolio happens to be 100% in bonds), and the "risky" part, (stock) is not as optimal as the growth optimal portfolio (unless the optimal growth portfolio happens to be 100% invested in stocks).

Our work can be extended in a number of directions. First, it would be desirable to introduce an integrated model of asset-liability management. The goal there would be to derive the optimal solution in a continuous-time fully integrated model, with contribution and borrowing decisions included, as well as their impact on optimal capital structure of the firm. Another possible extension would involve examining the impact of the presence of liability constraints from an equilibrium perspective. This would go along the lines of Basak (1995) study of the impact of portfolio insurance strategies on asset prices.

6. References

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8. Appendices

Proof of Proposition 2

Using Itô's lemma, and the processes for the return on the assets and liability give in (1) and (2), we have that: $\frac{d\hat{P}_t}{\hat{P}_t} = \frac{dP_t}{P_t} - \frac{dL_t}{L_t} - \operatorname{cov}_t\left(\frac{dP_t}{P_t}, \frac{dL_t}{L_t}\right) + Var_t\left(\frac{dL_t}{L_t}\right)$

This can also be written as

$$\frac{d\hat{P}_t}{\hat{P}_t} = (\sigma - \sigma_L I_n)(dW_t + (\theta - \sigma_L \mathbf{1})dt) - \sigma_{L,\varepsilon}(dW_t^{\varepsilon} + (\theta_L - \sigma_{L,\varepsilon})dt) \qquad (3)$$

Here I_n is defined as the *(nxn)* identity matrix. We can also re-write equation (3) as $\frac{d\hat{P}_t}{\hat{P}_t} = (\sigma - \sigma_L I_n) dW_t^{Q_L} - \sigma_{L,\varepsilon} dW_t^{Q_L,\varepsilon}$, with:

$$
\begin{aligned}
dW_t^{Q_L} &= dW_t + (\theta - \sigma_L \mathbf{1})dt \\
dW_t^{Q_L,\varepsilon} &= dW_t^{\varepsilon} + (\theta_L - \sigma_{L,\varepsilon}\mathbf{1})dt
\end{aligned}
$$

We now look for the measure under which prices are martingales when the liability value is used as a numeraire. Using Girsanov theorem, we know that $(W_t^{Q_L})_{t\geq 0}$ and $(W_t^{Q_L,\varepsilon})_{t\geq 0}$ are, respectively, a *n*-dimensional and a *1*-dimensional martingales under the probability Q_L with a Radon-Nikodym derivatives with respect to the original probability P defined as the product of the following two stochastic integrals:[11]

$$
\frac{dQ_L}{dP} = \xi(0,T)\xi_L(0,T)
$$

with:

$$
\begin{aligned}
\xi(0,T) &\equiv \exp\left\{-\int_0^T \kappa'(t) dW_t - \frac{1}{2}\int_0^T \kappa'(t)\kappa(t)dt\right\} \\
\xi_L(0,T) &\equiv \exp\left\{-\int_0^T \kappa_L(t) dW_t^{\varepsilon} - \frac{1}{2}\int_0^T \kappa_L^2(t)dt\right\}
\end{aligned}
$$

where $\kappa(t) = \theta(t) - \sigma_L(t)\mathbf{1}$, and $\kappa_L(t) = \theta_L(t) - \sigma_{L,\varepsilon}\mathbf{1}$.

Proof of Theorem 1

The Lagrangian for this problem is:

$$
L = E_t\left[\frac{\left(\frac{A_T}{L_t \eta(t,T)\eta_L(t,T)}\right)^{1-\gamma}}{1-\gamma}\right] - \lambda\left\{E_t\left[\xi(t,T)\xi_L(t,T)\frac{A_T}{L_t\eta(t,T)\eta_L(t,T)}\right] - \frac{A_t}{L_t}\right\}
$$

or:

[11] This is again because the Brownian motion driving pure liability risk uncertainty is orthogonal to the Brownian motion driving asset price uncertainty.

$$L=\frac{1}{(1-\gamma)(L_t)^{1-\gamma}}E_t\left[\left(\frac{A_T}{\eta(t,T)}\right)^{1-\gamma}\right]E_t\left[\frac{1}{(\eta_L(t,T))^{1-\gamma}}\right]-\lambda\left\{E_t\left[\frac{A_T\xi(t,T)}{\eta(t,T)}\right]\frac{1}{L_t}E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]-\frac{A_t}{L_t}\right\}$$

The first order conditions read:

$$\left(\frac{A_T^*}{L_t\eta(t,T)}\right)^{-\gamma}E_t\left[\frac{1}{(\eta_L(t,T))^{1-\gamma}}\right]=\lambda\xi(t,T)E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right] \qquad \text{(A-1)}$$

and:

$$E_t\left[\frac{A_T^*\xi(t,T)}{\eta(t,T)}\right]E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]=A_t \qquad \text{(A-2)}$$

From (A-1), we obtain:

$$A_T^*=L_t\eta(t,T)(\xi(t,T))^{-\frac{1}{\gamma}}\lambda^{-\frac{1}{\gamma}}\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]\right)^{-\frac{1}{\gamma}}\left(E_t\left[\frac{1}{(\eta_L(t,T))^{1-\gamma}}\right]\right)^{\frac{1}{\gamma}} \qquad \text{(A-3)}$$

Substituting (A-3) into (A-2), we solve for λ

$$\lambda=\left(\frac{A_t}{L_t}\right)^{-\gamma}\left[E_t\left((\xi(t,T))^{1-\frac{1}{\gamma}}\right)\right]^{\gamma}\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]\right)^{\gamma-1}\left(E_t\left[\frac{1}{(\eta_L(t,T))^{1-\gamma}}\right]\right) \qquad \text{(A-4)}$$

which we substitute back into (A-3) to obtain the optimal terminal asset value :

$$A_T^*=A_t\eta(t,T)(\xi(t,T))^{-\frac{1}{\gamma}}\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]\right)^{-1}\left(E_t[\xi(t,T)]^{1-\frac{1}{\gamma}}\right)^{-1}$$

or equivalently in terms of optimal terminal funding ratio:

$$F_T^*=\frac{A_T^*}{L_t\eta(t,T)\eta_L(t,T)}=\frac{A_t}{L_t}(\eta_L(t,T))^{-1}(\xi(t,T))^{-\frac{1}{\gamma}}\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]\right)^{-1}\left(E_t[\xi(t,T)]^{1-\frac{1}{\gamma}}\right)^{-1} \qquad \text{(A-5)}$$

The indirect utility function is: $J_t=J(t,F_t)=E_t\left[\frac{(F_T^*)^{1-\gamma}}{1-\gamma}\right]$. Substituting F_T^* from (A-5) in this equation, we get:

$$J_t=\frac{(F_t)^{1-\gamma}}{1-\gamma}\left(E_t\left[(\xi(t,T))^{1-\frac{1}{\gamma}}\right]\right)^{\gamma}\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)}\right]\right)^{\gamma-1}E_t\left[(\eta_L(t,T))^{\gamma-1}\right]$$

We therefore obtain that the indirect utility function is separable in the funding ratio: $J_t = \frac{(F_t)^{1-\gamma}}{1-\gamma} g(t,T)$, with: $g(t,T) = \left(E_t\left[(\xi(t,T))^{1-\frac{1}{\gamma}} \right] \right)^{\gamma} \left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)} \right] \right)^{\gamma-1} E_t\left[(\eta_L(t,T))^{\gamma-1} \right]$.

Standard calculation of expectation of an exponential of a Gaussian variable gives the following results (in the case of constant parameters):

$$\left(E_t\left[(\xi(t,T))^{1-\frac{1}{\gamma}} \right] \right)^{\gamma} = \exp\left[-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)\kappa'\kappa(T-t) \right]$$

$$\left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)} \right] \right)^{\gamma-1} = \exp\left[-\gamma\left(\sigma_{L,\varepsilon}^2 - \kappa_L \sigma_{L,\varepsilon}\right)(T-t) \right]$$

$$E_t\left[(\eta_L(t,T))^{\gamma-1} \right] = \exp\left[\frac{(1-\gamma)(2-\gamma)}{2}\sigma_{L,\varepsilon}^2 (T-t) \right]$$

To derive the optimal portfolio strategy, we introduce the following notation for any $t<T$, and for $s>t$:

$$G_s = E_s^{Q_L}\left(F_T^*\right) = E_s\left[\xi(s,T)\xi_L(s,T)F_T^* \right]$$

Using equation (A-5), we have that:

$$G_s = E_s^{Q_L}\left(F_T^*\right) = E_s\left[\xi(s,T)\xi_L(s,T)(\eta_L(t,T))^{-1} \frac{A_t}{L_t} (\xi(t,T))^{-\frac{1}{\gamma}} \left(E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)} \right] \right)^{-1} \left(E_t\left[\xi(t,T) \right]^{1-\frac{1}{\gamma}} \right)^{-1} \right]$$

or:

$$G_s = \frac{F_t}{E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)} \right] E_t\left(\left[\xi(t,T) \right]^{1-\frac{1}{\gamma}} \right)} (\eta_L(t,s))^{-1} (\xi(t,s))^{-\frac{1}{\gamma}} E_s\left[\xi(s,T)\xi_L(s,T)(\eta_L(s,T))^{-1} (\xi(s,T))^{-\frac{1}{\gamma}} \right]$$

which finally translates into:

$$G_s = F_t \frac{E_s\left(\left[\xi(s,T) \right]^{1-\frac{1}{\gamma}} \right)}{E_t\left(\left[\xi(t,T) \right]^{1-\frac{1}{\gamma}} \right)} \frac{E_s\left[\frac{\xi_L(s,T)}{\eta_L(s,T)} \right]}{E_t\left[\frac{\xi_L(t,T)}{\eta_L(t,T)} \right]} \frac{(\xi(t,s))^{-\frac{1}{\gamma}}}{(\eta_L(t,s))}$$

The stochastic terms in G_s come from the stochastic terms in $\frac{(\xi(t,s))^{-\frac{1}{\gamma}}}{(\eta_L(t,s))}$. Using Ito's lemma, we obtain:

$$\frac{dG_s}{G_s} = \mu_G ds + \frac{1}{\gamma}\kappa^{'} dW_s - \sigma_{L,\varepsilon} dW_s^{\varepsilon}$$

Beside we had that :

$$\frac{dF_s^w}{F_s^w} = \left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\varepsilon}^2\right)ds + w'\left((\mu - r\mathbf{1}) - \sigma\sigma_L\right)ds + \left(w'\sigma - \sigma_L^{'}\right)dW_s - \sigma_{L,\varepsilon} dW_s^{\varepsilon}$$

Identifying the terms in front of the multi-dimensional Brownian motion driving asset prices dW_s, we have that $\left(w'\sigma - \sigma_L^{'}\right) = \frac{1}{\gamma}\kappa^{'}$, an equation we can solve for the optimal $w = \frac{1}{\gamma}\sigma^{-1}\kappa + (\sigma')^{-1}\sigma_L$.

Proof of Theorem 2 (Complete Market Case)

The Lagrangian for this problem is:

$$L = E_t\left[\frac{(F_T - k)^{1-\gamma}}{1-\gamma}\right] - \lambda\left\{E_t\left[\xi(t,T)F_T\right] - F_t\right\}$$

The first order conditions read:

$$\left(F_T^* - k\right)^{-\gamma} = \lambda\xi(t,T) \qquad \text{(B-1)}$$

$$E_t\left[F_T^*\xi(t,T)\right] = F_t \qquad \text{(B-2)}$$

From (B-1), we obtain:

$$F_T^* = \left(\xi(t,T)\right)^{-\frac{1}{\gamma}}\lambda^{-\frac{1}{\gamma}} + k \qquad \text{(B-3)}$$

Hence the horizon funding ratio equals that of the unconstrained case plus a riskless amount *k*.

Substituting (B-3) into (B-2), we solve for λ.

$$\lambda = \left(F_t\right)^{-\gamma}\left[E_t\left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T)\right)\right]^{\gamma}$$

which we substitute back into (B-3) to obtain the optimal terminal asset value :

$$F_T^* = F_t \left[E_t \left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T) \right) \right]^{-1} \left(\xi(t,T)\right)^{-\frac{1}{\gamma}} + k \qquad \text{(C-4)}$$

The indirect utility function is:

$$J_t = J(t, F_t) = E_t \left[\frac{\left(F_T^* - k\right)^{1-\gamma}}{1-\gamma} \right]$$

Substituting F_T^* from (B-4) in this equation, we get:

$$J_t = \frac{1}{1-\gamma} \left(F_t \left[E_t \left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T) \right) \right]^{-1} \left(\xi(t,T)\right)^{-\frac{1}{\gamma}} \right)^{1-\gamma}$$

To derive the optimal portfolio strategy, note that for any $t<T$, and for $s>t$ we have that: $F_s = E_s^{Q_L}\left(F_T^*\right) = E_s\left[\xi(s,T) F_T^*\right]$. Using equation (B-4), we have that:

$$F_s = E_s^{Q_L}\left(F_T^*\right) = E_s \left[\xi(s,T) F_t \left[E_t \left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T) \right) \right]^{-1} \left(\xi(t,T)\right)^{-\frac{1}{\gamma}} + k\xi(s,T) \right]$$

or:

$$F_s = \frac{F_t}{E_t \left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T) \right)} E_s \left[\xi(s,T) \left(\xi(t,T)\right)^{-\frac{1}{\gamma}} \right] + k E_s\left[\xi(s,T)\right]$$

or also (since $E_s\left[\xi(s,T)\right] = 1$)

$$F_s = \frac{F_t}{E_t \left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T) \right)} E_s \left[\xi(s,T) \left(\xi(t,T)\right)^{-\frac{1}{\gamma}} \right] + k$$

or again :

$$F_s = \frac{F_t}{E_t \left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T) \right)} E_s \left[\left(\xi(s,T)\right)^{1-\frac{1}{\gamma}} \right] \left(\xi(t,s)\right)^{-\frac{1}{\gamma}} + k$$

The stochastic terms in F_s come from the stochastic terms in $\left(\xi(t,s)\right)^{-\frac{1}{\gamma}}$. Using Ito's lemma, we obtain:

$$dF_s = \frac{F_t}{E_t\left(\left(\xi(t,T)\right)^{1-\frac{1}{\gamma}} + k\xi(t,T)\right)} E_s\left[\left(\xi(s,T)\right)^{1-\frac{1}{\gamma}}\right]\frac{1}{\gamma}\kappa^{'} dW_s + (...)ds$$

which is also equal to:

$$dF_s = (F_s - k)\frac{1}{\gamma}\kappa^{'} dW_s + (...)ds$$

Finally, we obtain: $\frac{dF_s}{F_s} = \left(1 - \frac{k}{F_s}\right)\frac{1}{\gamma}\kappa^{'} dW_s + (...)ds$. Beside we had that :

$$\frac{dF_s^w}{F_s^w} = \left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\varepsilon}^2\right)ds + w'\left((\mu - r\mathbf{1}) - \sigma\sigma_L\right)ds + \left(w'\sigma - \sigma_L^{'}\right)dW_s - \sigma_{L,\varepsilon} dW_s^{\varepsilon}$$

Identifying the terms in front of the multi-dimensional Brownian motion driving asset prices dW_s, we have that $\left(w'\sigma - \sigma_L^{'}\right) = \left(1 - \frac{k}{F_s}\right)\frac{1}{\gamma}\kappa^{'}$, an equation we can solve for the optimal portfolio weights:

$$w^* = \frac{1}{\gamma}\left(1 - \frac{k}{F_s}\right)(\sigma\sigma')^{-1}(\mu - r1) + \left(1 - \frac{1}{\gamma}\left(1 - \frac{k}{F_s}\right)\right)(\sigma')^{-1}\sigma_L$$