

# Optimal Policy with Low-Probability Extreme Events

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## Introduction

A standard result in optimal-control theory is *certainty equivalence*, which results under the assumption of a linear model with additive uncertainty and a quadratic loss function. Certainty equivalence implies that only the *mean* values (that is, the probability-weighted average outcomes) of target variables matter for the optimal policy setting. Consider monetary policy under flexible inflation targeting, where inflation and the output gap are the target variables. Because of the sizable lags in the full effects of monetary policy actions on output and inflation, say about a year for output and about two years for inflation, such monetary policy is most effective if it is forward looking and relies on inflation and output-gap projections. Certainty equivalence then implies that optimal monetary policy need only consider mean inflation and output-gap projections. That is, the central bank need only to do flexible “mean” inflation targeting, regardless of the degree of uncertainty of the projections. Only the first (statistical) moment matters for policy, not the higher moments.

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Furthermore, certainty equivalence implies that a *low-probability extreme event* (that is, a large shock with a small probability) should be taken into account by the *mean size* of the event, that is, the probability-weighted size of the shock. Such an event could, for instance, be a large and sudden depreciation of the currency (say, the pound or the dollar), a bursting of an asset-price bubble (say, a property-price bubble), or a war in the Middle East with, among other disturbances, an associated oil-price hike. For the Bank of England, the possibility of a sudden substantial depreciation of the pound and resulting higher inflation has been a relevant example. If the Bank of England normally aims at achieving the inflation target at a two-year horizon, this would imply setting interest rates such that mean inflation about two years ahead, including the possibility of a sudden depreciation, approximately equals the inflation target. If a depreciation of the pound would not occur during the next two years, actual inflation would most likely end up lower than the inflation target. Since a depreciation would occur only by a small (subjective) probability, actual inflation would with a large probability fall short of the inflation target.

In informal discussions, several central bankers have questioned whether, in such situations, it may make sense to put less weight on the low-probability event than suggested by certainty equivalence and the mean size of the event, perhaps even to ignore the low-probability event altogether. For instance, another policy option is to wait and see, and to deal with the event only if it occurs. Freedman (2002), at the Jackson Hole Symposium in 2002, sponsored by the Federal Reserve Bank of Kansas City, stated, in commenting on Svensson (2002):

There are two points about this paper I'd like to make. The first relates to the mean versus mode debate, and I'd like to tie that to the asset price issue. One of the real challenges to central banks is how to deal with small probability cases. If we have a situation where there is, say, a 10 percent probability of an asset price collapse and a 90 percent probability that it is not going to happen, do you then go ahead and focus on what one would call the mode, which is the appropriate path for policy in the 90 percent case and say, "If the 10 percent case happens, we'll try and deal with it later."? Or do we try to deal with the mean? In that case if the small probability outcome doesn't happen, you are going to have an interest rate path that is not appropriate. But even if it does happen, you will not have moved interest rates enough to deal with the collapse of asset prices in any case. So, it is very much an open question of how to deal with such a situation.

The Monetary Policy Committee (MPC) of the Bank of England has raised issues like these, for instance, on 6–7 February 2002, in discussing the issue of a potential depreciation of the pound (Bank of England 2002, 10):

[S]ome members placed weight on upside risks to the inflation outlook. Two main risks to inflation were emphasised: from the possibility of a depreciation of sterling's exchange rate and from the possibility that consumption would not slow as much as projected. . . . There was evidence that the exchange rate was overvalued. . . . Other members placed less weight on the exchange rate risk in the published fan charts—either because they thought the probability of a depreciation was low or because the pass through to prices would be small or because policy could react if and when sterling fell.

Furthermore, MPC member Nickell (2002, 19) discusses the issue of exchange rate movements for the Bank of England's monetary policy in somewhat greater detail:

The important question here is should we move interest rates up today in order to forestall the potential inflationary consequences of a significant exchange rate fall which might come about as UK domestic demand growth slows and international demand growth speeds up?

He prefers a wait-and-see approach, though:

In my view, the answer is simply no. If, and when, such a fall in the exchange rate comes about, then is the time to make appropriate adjustments, if any, in interest rates. To act preemptively on this front is difficult because of the huge uncertainties involved in forecasting exchange rate movements and not really necessary because the lags to demand from interest rates and exchange rates are of the same order of magnitude.

He clarifies the last point by stating (Nickell 2002, footnote 16):

The argument here is that the first round, price level effects from sudden exchange rate moves should be accommodated, with monetary policy only acting on the potential second-round effects.

The quotes from the Bank of England's minutes and Nickell's speech could be interpreted as indicating a desire to assume the problem away and to argue that there is no trade-off, because (i) it is appropriate to accommodate the first-round effect on inflation of a currency depreciation, and (ii) the

second-round effects on inflation have such long lags that they can be handled by interest rate adjustments once the depreciation has occurred. But what if it would be undesirable to accommodate also the first-round effect on the price level?

Clearly, if the extreme event in question is such that there is time to react to it in case it materializes, there is no point in trying to pre-empt the event. Instead, there is an option value of waiting to see if the event materializes and only then take the appropriate action. But assume that the event will, with some probability, occur within the lag between a policy action and its impact on the target variables and, furthermore, have a sizable direct impact on the target variable within that lag, so that any policy action after the event cannot undo this. Then, there is a genuine trade-off between acting to pre-empt the event by taking it into account by its mean size, and not pre-empting.

This paper tries to come to grips with problems of this type, when there is a genuine trade-off between pre-empting and not, by examining what kind of optimal policy for a low-probability shock is implied by other loss functions than the standard quadratic one.<sup>1</sup> The paper considers a simple policy problem with a sequence of different loss functions: quadratic, combined quadratic/absolute-deviation, absolute-deviation, combined quadratic/constant, and perfectionist.<sup>2</sup> Obviously, this is a short and far from exhaustive list of possible loss functions. The paper shows that, under some simplifying assumptions, each of these loss functions puts less weight on a low-probability extreme event than the previous one, down to the quadratic/constant and perfectionist loss functions, which completely ignore the low-probability extreme event. The paper also examines the case when the size of the extreme shock is endogenous and depends on the policy instrument. This introduces an additional effect on optimal policy of the extreme shock for all loss functions except the combined quadratic/constant and the perfectionist.

## 1 A Simple Policy Problem

Let  $\pi$  denote (the rate of) future inflation, and assume that future inflation is determined by the current state of the economy,  $x$ , a current (policy) instrument,  $i$ , and an exogenous future stochastic shock,  $z$ , according to

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1. Vickers (1998) and Wallis (1999) have previously examined whether some of these loss functions motivate mean, median, or mode inflation targeting under normal circumstances.
  2. The combined quadratic/constant loss function can be seen as a simplification of the bell loss function examined by Bray and Goodhart (2002).

$$\pi = x - \alpha i + z,$$

where  $\alpha$  is a positive constant. Assume that the instrument has to be set before the future shock is known but after the current state of the economy is observed. Assume that the future shock is the sum of two independently distributed random variables,  $\varepsilon$  and  $\eta$ ,

$$z = \varepsilon + \eta.$$

Assume that  $\varepsilon$  has a zero mean and a continuous distribution. It will be interpreted as a future normal-size shock with relatively modest variance. Let  $f(\varepsilon)$  denote the density function of the normal-size shock.

Let  $\eta$  have a discrete distribution such that it can take two values, 0 and  $a$ , with probability  $1 - \gamma > 0$  and  $\gamma > 0$ , respectively. Let  $a > 0$  be a given large number, and let  $\gamma$  be a small probability. (The case when the size of the shock is endogenous will be examined in section 7.) The high-probability outcome,  $\eta = 0$ , and the resulting shock,  $z = \varepsilon$ , will be identified with a future *normal* outcome. The low-probability outcome,  $\eta = a$ , and the resulting shock,  $z = a + \varepsilon$ , will be identified with a future *extreme* outcome. Thus, the extreme outcome is here a large inflationary shock.<sup>3</sup>

Let  $h(z)$  be the density function of the future shock  $z$ . Under the assumptions above, it will be given by

$$h(z) \equiv (1 - \gamma)f(z) + \gamma f(z - a). \quad (1)$$

An example of such a density function is shown in Figure 1. The *unconditional mean* of  $z$ ,  $Ez$ , fulfills

$$Ez \equiv \int z h(z) dz = \gamma a.$$

For future reference, let  $\text{med}(z)$  denote the *median* of  $z$ . It is (for a continuous distribution) defined by

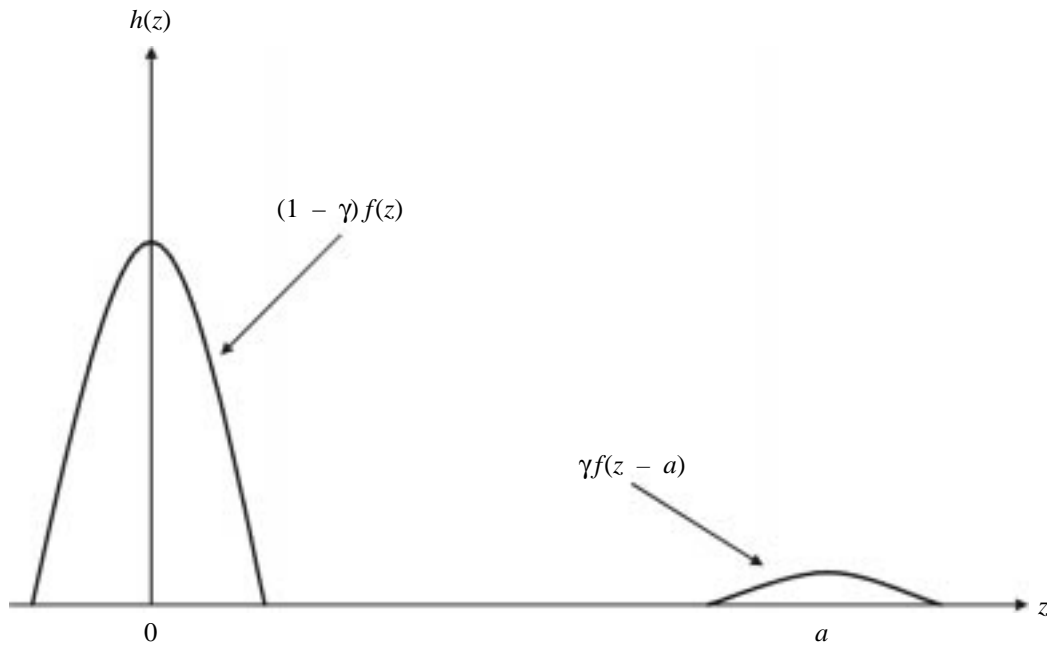
$$\Pr[z \leq \text{med}(z)] = \frac{1}{2}, \quad (2)$$

where  $\Pr[z \leq y] \equiv \int_{-\infty}^y h(z) dz$  denotes the probability that the realization of the future shock  $z$  is less than or equal to a given value  $y$ . Furthermore, let

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3. The extreme outcome could be a large deflationary shock. This brings in issues of the zero lower bound on interest rates and the risk for the economy of falling into a liquidity trap with a prolonged recession and deflation. These topical issues have to do with nonlinearities in the transmission mechanism of monetary policy and raise separate issues that are not discussed here. See Svensson (2003a), for instance, for a recent discussion of deflation and liquidity traps and references to the literature on those issues.

**Figure 1**  
**The density function of the shock**



mode ( $z$ ) denote the *mode* of  $z$ , that is, the set of  $z$  (which can have several elements) for which the probability density is the highest,

$$\text{mode}(z) \equiv \arg \max h(z).$$

Let  $\bar{\pi}$  denote expected future inflation conditional on the future normal outcome, the *normal mean (future) inflation*,

$$\bar{\pi} \equiv E[\pi; \eta = 0].$$

We note that normal mean inflation is related to the current state of the economy and the current instrument by

$$\bar{\pi} = x - \alpha i, \quad (3)$$

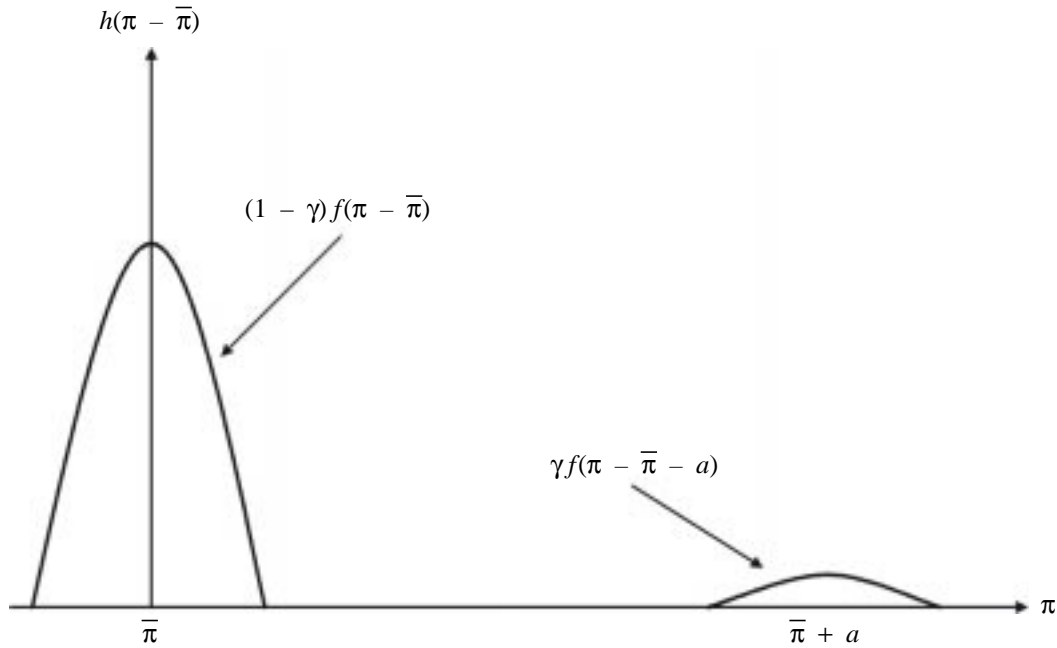
and that the relation between realized future inflation and normal mean inflation is

$$\pi = \bar{\pi} + z. \quad (4)$$

For given normal mean inflation, the *density function of future inflation* is then given by  $h(\pi - \bar{\pi})$ . It is shown in Figure 2. *Unconditional mean (future) inflation*,  $E\pi$ , will fulfill

$$E\pi \equiv \int \pi h(\pi - \bar{\pi}) d\bar{\pi} = \bar{\pi} + \gamma a. \quad (5)$$

**Figure 2**  
**The density function of future inflation**



It follows that we can conveniently discuss the policy action in terms of the resulting normal mean inflation, and for given normal mean inflation (and current state of the economy), then infer the corresponding instrument setting from equation (3).

Let  $L(\pi)$  be a loss function over inflation with a fixed inflation target,  $\pi^*$ , which we take to be a low positive number. The loss function has a minimum for the inflation target:  $L(\pi) \geq L(\pi^*)$  for all  $\pi$ . The minimum will be unique for the loss functions considered here ( $L(\pi) > L(\pi^*)$  for all  $\pi \neq \pi^*$ ), but this is not necessary and the treatment here can easily be extended to loss functions that have non-unique minima, for instance those corresponding to a “zone of indifference.”<sup>4</sup>

The policy problem then consists of selecting the normal mean inflation,  $\bar{\pi}$ , so as to minimize the *expected loss*,

$$EL(\pi) \equiv \int L(\pi)h(\pi - \bar{\pi})d\pi = \int L(\bar{\pi} + z)h(z)dz.$$

4. A loss function fulfilling  $L(\pi) = 0$  for  $|\pi - \pi^*| \leq c > 0$  and  $L(\pi) = c$  for  $|\pi - \pi^*| > c$  is an example of a zone-of-indifference loss function.

Since inflation is the only target variable (the only variable entering the loss function), this is a case of so-called “strict” inflation targeting—an unrealistic but simple case to deal with.

The first-order condition for an optimum can then be written

$$\frac{\partial}{\partial \bar{\pi}} E L(\pi) = \int \frac{\partial L(\bar{\pi} + z)}{\partial \pi} h(z) dz = \int \frac{\partial L(\pi)}{\partial \pi} h(\pi - \bar{\pi}) d\pi = 0,$$

where  $\frac{\partial L(\pi)}{\partial \pi}$  is the *marginal loss* (function), the derivative of the loss function with respect to inflation. Hence, we can write this simply as

$$E \frac{\partial L(\pi)}{\partial \pi} = 0. \quad (6)$$

That is, the normal mean inflation should be chosen so that the *expected marginal loss* equals zero. This is the *optimal targeting rule* for this problem (see, for instance, Svensson (2003b) on targeting rules).

## 2 A Quadratic Loss Function: Mean Inflation Targeting

Consider a quadratic loss function,

$$L(\pi) \equiv \frac{1}{2}(\pi - \pi^*)^2. \quad (7)$$

This loss function is shown as the solid curve in Figure 3. The marginal loss is continuous, linear, and increasing in inflation,

$$\frac{\partial L(\pi)}{\partial \pi} = \pi - \pi^*. \quad (8)$$

It is shown as the solid line AA' in Figure 4.

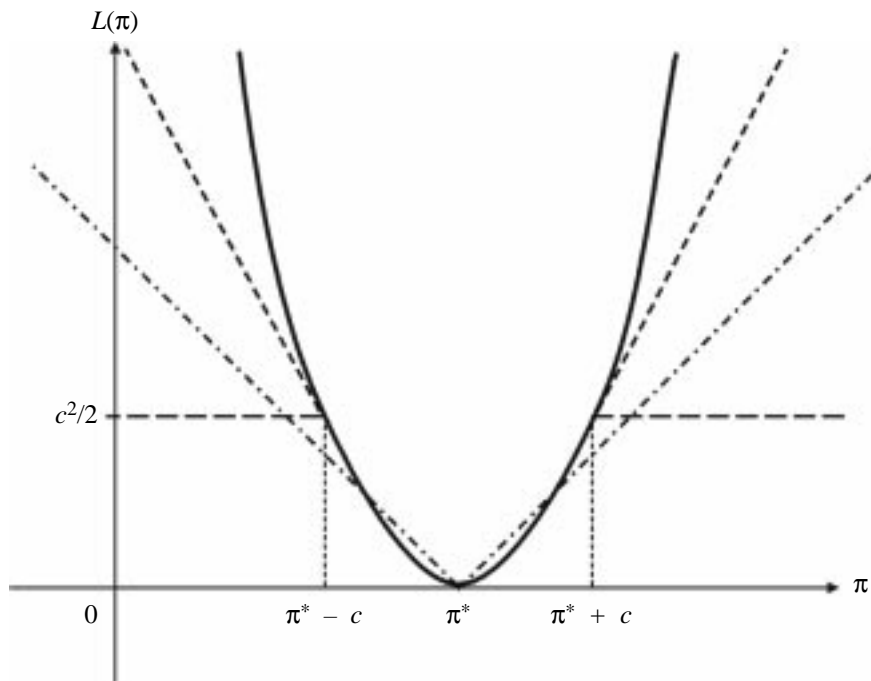
From the optimal targeting rule, equation (6), and the marginal loss, equation (8), it follows that the optimal targeting rule in the quadratic case is simply

$$E\pi = \pi^*.$$

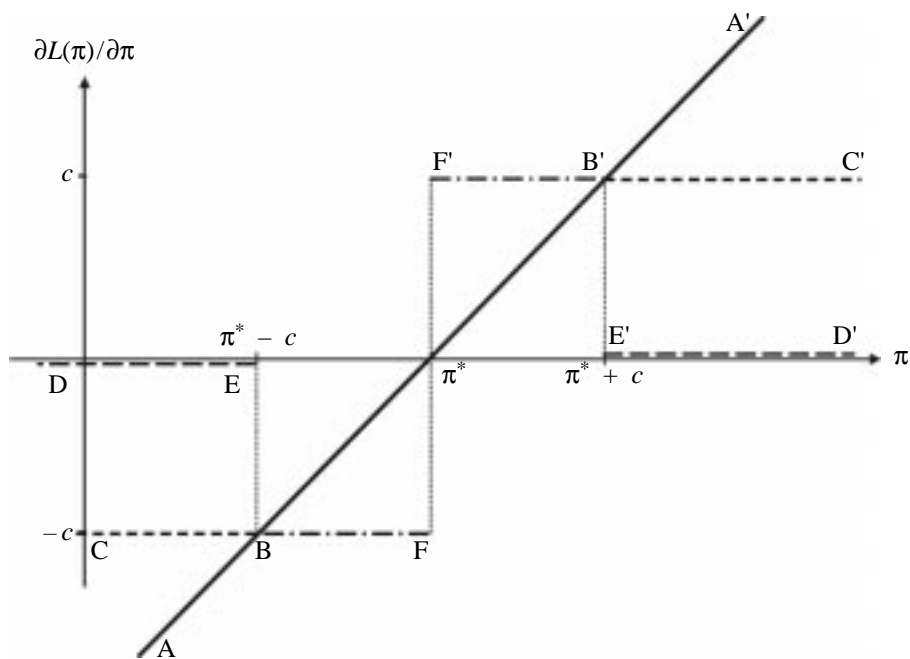
That is, normal mean inflation should be set so as to make unconditional mean inflation equal to the inflation target, what we can call (unconditional) *mean inflation targeting*. This is an example of the certainty equivalence that



**Figure 3**  
Alternative loss functions



**Figure 4**  
Alternative marginal loss functions



holds under the assumption of a linear model with additive uncertainty and a quadratic loss function.

It follows from equation (5) that the optimal normal mean inflation with a quadratic loss function, denoted  $\bar{\pi}^q$ , is given by

$$\bar{\pi}^q = \pi^* - \gamma a.$$

It hence undershoots the inflation target by the mean future extreme event,  $\gamma a$ , the probability-weighted size of the future extreme shock.

### 3 An Absolute-Deviation Loss Function: Median Inflation Targeting

Consider an absolute-deviation loss function, that is, a loss function that is linear in the absolute deviation of inflation from the inflation target,

$$L(\pi) \equiv |\pi - \pi^*|. \quad (9)$$

This loss function is shown as the V-shaped dashed-dotted line in Figure 3. For this loss function, the marginal loss is discontinuous at the inflation target and otherwise constant,

$$\frac{\partial L(\pi)}{\partial \pi} = \begin{cases} -1 & \text{for } \pi < \pi^*, \\ 1 & \text{for } \pi > \pi^*. \end{cases}$$

It is shown as the step function CBFF'B'C' in Figure 4.

The expected marginal loss is hence

$$\begin{aligned} E \frac{\partial L(\pi)}{\partial \pi} &= - \int_{-\infty}^{\pi^*} h(\pi - \bar{\pi}) d\pi + \int_{\pi^*}^{\infty} h(\pi - \bar{\pi}) d\pi \\ &= - \Pr[\pi \leq \pi^*] + \Pr[\pi > \pi^*] \\ &= 1 - 2\Pr[\pi \leq \pi^*], \end{aligned}$$

where I have used that  $\Pr[\pi > \pi^*] = 1 - \Pr[\pi \leq \pi^*]$ . The optimal targeting rule, equation (6), requires setting this equal to zero, which results in

$$\Pr[\pi \leq \pi^*] = \frac{1}{2}. \quad (10)$$

It follows from equation (2) that normal mean inflation should be set so that median future inflation equals the inflation target, what we can call *median inflation targeting*,

$$\text{med}(\pi) = \pi^*. \quad (11)$$

It then follows from equations (4) and (11) that the optimal normal mean inflation with an absolute-deviation loss function, denoted  $\bar{\pi}^a$ , is given by

$$\bar{\pi}^a = \pi^* - \text{med}(z). \quad (12)$$

### 3.1 A special case

Assume that the normal future shock,  $\varepsilon$ , has a bounded symmetric support,  $\varepsilon \in [-b, b]$  (where  $b > 0$  is a constant).<sup>5</sup> Furthermore, assume that

$$a > 2b. \quad (13)$$

This implies that the support of the extreme shock,  $z = a + \varepsilon \in [a - b, a + b]$ , falls outside the support of the normal shock (this is the case for the density function shown in Figure 1). This is consistent with the previous assumption that the variance of the normal shock  $\varepsilon$  is modest relative to the size of the extreme event.

Furthermore, in order to conveniently compute the median of the total future shock,  $z$ , assume that  $\varepsilon$  is uniformly distributed,

$$f(\varepsilon) = \begin{cases} \frac{1}{2b} & \text{for } |\varepsilon| \leq b, \\ 0 & \text{for } |\varepsilon| > b. \end{cases} \quad (14)$$

Then the density function of the future shock,  $h(z)$ , is given by

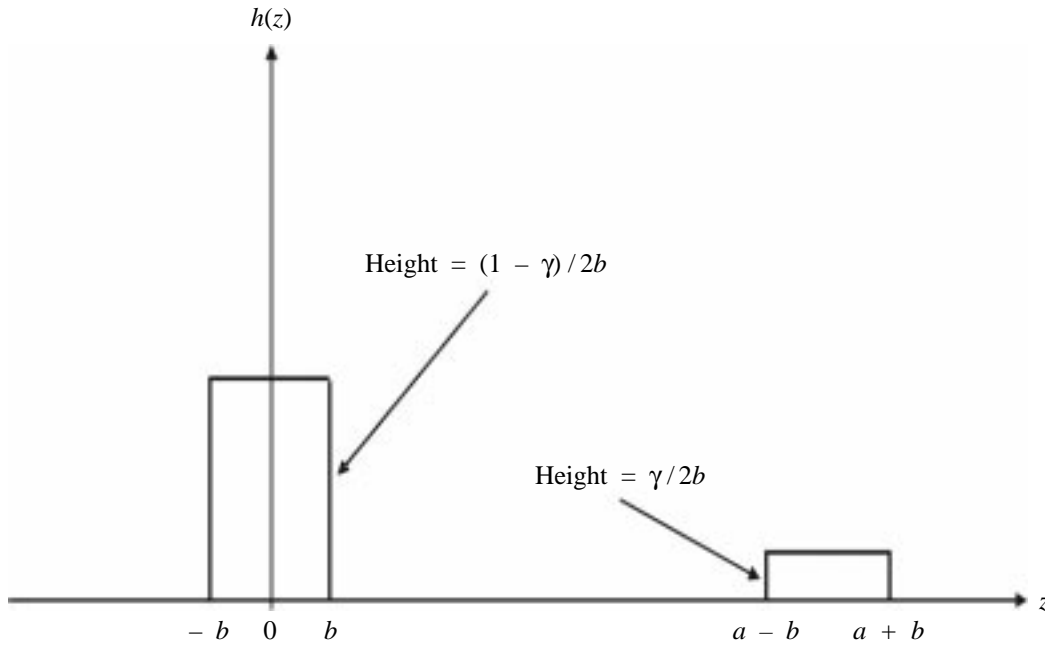
$$h(z) = \begin{cases} 0 & \text{for } z < -b, \\ (1 - \gamma)\frac{1}{2b} & \text{for } -b \leq z \leq b, \\ 0 & \text{for } b < z < a - b, \\ \gamma\frac{1}{2b} & \text{for } a - b \leq z \leq a + b, \\ 0 & \text{for } a + b < z. \end{cases} \quad (15)$$

This density function is illustrated in Figure 5.

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5. The support of a probability distribution is the set of outcomes for which the density or probability is positive.

**Figure 5**  
**The special-case density function of the shock**



To compute the median of  $z$ ,  $\text{med}(z)$ , we assume that  $\gamma < \frac{1}{2}$  (recall that we assume that  $\gamma$  is a small positive number) and use that the median then falls within the support of the normal shock,

$$0 < \text{med}(z) < b. \quad (16)$$

It follows from equation (15) that

$$\Pr[z \leq \text{med}(z)] = (1 - \gamma) \frac{1}{2b} [\text{med}(z) + b],$$

which, together with equation (2), results in a convenient expression for the median of  $z$ ,

$$\text{med}(z) = \frac{\gamma}{1 - \gamma} b, \quad (17)$$

confirming inequality (16). It follows from equations (12) and (17) that normal inflation with an absolute-deviation loss function will fulfill

$$\bar{\pi}^a = \pi^* - \frac{\gamma}{1 - \gamma} b < \pi^*. \quad (18)$$

Unconditional mean inflation will then fulfill

$$E\pi = \pi^* + \gamma \left( a - \frac{1}{1-\gamma} b \right) > \pi^*,$$

where the inequality holds for  $\gamma < \frac{1}{2}$  and inequality (13).

#### 4 A Combined Quadratic/Absolute-Deviation Loss Function: Less Weight on the Extreme Event

Consider a loss function that is quadratic for moderate deviations of inflation from the inflation target but linear in the absolute deviation from the target for large deviations,

$$L(\pi) \equiv \begin{cases} \frac{1}{2}(\pi - \pi^*)^2 & \text{for } |\pi - \pi^*| \leq c, \\ c|\pi - \pi^*| - \frac{1}{2}c^2 & \text{for } |\pi - \pi^*| > c, \end{cases} \quad (19)$$

where  $c > 0$  denotes the deviation from the target at which the loss function shifts from quadratic to linear. I shall refer to  $|\pi - \pi^*| \leq c$  as the *quadratic range* of the loss function and to  $|\pi - \pi^*| > c$  as the *linear range*. This loss function is shown as the combined solid curve for the quadratic range and short-dashed lines for the linear range in Figure 3.

Such a loss function represents a situation when a quadratic loss function with a rising marginal loss is considered appropriate for moderate deviations from the target. For large deviations from the target, the marginal loss is considered constant rather than increasing, corresponding to a linear range of the loss function.

The loss function is continuous and has a continuous marginal loss,

$$\frac{\partial L(\pi)}{\partial \pi} = \begin{cases} -c & \text{for } \pi - \pi^* < -c, \\ \pi - \pi^* & \text{for } |\pi - \pi^*| \leq c, \\ c & \text{for } \pi - \pi^* > c. \end{cases}$$

This marginal loss is shown in Figure 4 and consists of the constant segment CB, the increasing linear segment BB', and the constant segment B'C'.

The expected marginal loss is then

$$\begin{aligned}
E\frac{\partial L(\pi)}{\partial \pi} &= -\int_{-\infty}^{\pi^*-c} ch(\pi - \bar{\pi})d\pi + \int_{\pi^*-c}^{\pi^*+c} (\pi - \pi^*)h(\pi - \bar{\pi})d\pi + \int_{\pi^*+c}^{\infty} ch(\pi - \bar{\pi})d\pi \\
&= -c\Pr[\pi \leq \pi^* - c] + \int_{\pi^*-c}^{\pi^*+c} (\pi - \pi^*)h(\pi - \bar{\pi})d\pi + c\Pr[\pi > \pi^* + c] \\
&= -c\Pr[\pi - \pi^* \leq -c] + E[\pi - \pi^*; |\pi - \pi^*| \leq c]\Pr[|\pi - \pi^*| \leq c] \\
&\quad + c\Pr[\pi - \pi^* > c].
\end{aligned}$$

Setting the expected marginal loss equal to zero in accordance with the optimal targeting rule, equation (6), results in

$$E[\pi; |\pi - \pi^*| \leq c] = \pi^* - c \left\{ \frac{\Pr[\pi - \pi^* > c] - \Pr[\pi - \pi^* \leq -c]}{\Pr[|\pi - \pi^*| \leq c]} \right\}. \quad (20)$$

This condition says that mean inflation conditional on inflation falling within the quadratic range of the loss function should undershoot the inflation target by the product of the marginal loss in the linear range times the ratio between (i) the difference between the probability of inflation falling in the upper linear range and the probability of inflation falling in the lower linear range, and (ii) the probability of inflation falling in the quadratic range.

Regarding the special case discussed above, assume that  $a$ ,  $b$ , and  $c$  fulfill

$$a - b > \frac{1}{1 - \gamma}c > \frac{1 - 2\gamma}{1 - \gamma}c > b. \quad (21)$$

That is,  $a - b$  exceeds  $b$  in inequality (13) with a sufficient margin, so

$$\frac{1}{1 - \gamma}c \text{ and } \frac{1 - 2\gamma}{1 - \gamma}c$$

fit in between (recall that  $\gamma$  is a small number, so these two expressions are close to  $c$ ). This will imply that the normal support of inflation,  $[\bar{\pi} - b, \bar{\pi} + b]$ , will fall within the quadratic range, and the extreme support of inflation,  $[\bar{\pi} + a - b, \bar{\pi} + a + b]$ , will fall within the linear range of the loss function.

Given inequality (21), we then have

$$\begin{aligned}
E[\pi; |\pi - \pi^*| \leq c] &= \bar{\pi}, \Pr[\pi - \pi^* \leq -c] = 0 \text{ and} \\
\Pr[\pi - \pi^* > c] &= \gamma.
\end{aligned}$$

It follows from equation (20) that the optimal normal mean inflation under the quadratic/absolute-deviation loss function, denoted  $\bar{\pi}^{qa}$ , undershoots the

inflation target by  $\frac{\gamma}{1-\gamma}$  times the marginal cost of inflation in the linear range,<sup>6</sup>

$$\bar{\pi}^{qa} = \pi^* - \frac{\gamma}{1-\gamma}c < \pi^*. \quad (22)$$

Unconditional mean inflation then fulfills

$$E\pi = \pi^* + \gamma\left(a - \frac{1}{1-\gamma}c\right) > \pi^*,$$

where the inequality follows from inequality (21).

## 5 A Combined Quadratic/Constant Loss Function: Normal Mean Inflation Targeting

Consider a combined quadratic/constant loss function, that is, quadratic for moderate deviations of inflation from the inflation target and constant for large deviations,

$$L(\pi) \equiv \begin{cases} \frac{1}{2}(\pi - \pi^*)^2 & \text{for } |\pi - \pi^*| \leq c, \\ \frac{1}{2}c^2 & \text{for } |\pi - \pi^*| > c, \end{cases} \quad (23)$$

where  $c > 0$  denotes the deviation from the target at which the loss function shifts from quadratic to constant. The loss function is shown as the combined curve for the quadratic range and horizontal long-dashed line for the linear range in Figure 3. The loss function is continuous, whereas the marginal loss is discontinuous at  $\pi = \pi^* \pm c$ ,

$$\frac{\partial L(\pi)}{\partial \pi} = \begin{cases} \pi - \pi^* & \text{for } |\pi - \pi^*| < c, \\ 0 & \text{for } |\pi - \pi^*| > c. \end{cases}$$

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6. With this optimal normal mean inflation, the normal support of inflation is  $[\pi^* - \gamma c / (1 - \gamma) - b, \pi^* - \gamma c / (1 - \gamma) + b]$ , which by equation (21) falls within the quadratic range,  $[\pi^* - c, \pi^* + c]$ . The extreme support of inflation is  $[\pi^* - \gamma c / (1 - \gamma) + a - b, \pi^* - \gamma c / (1 - \gamma) + a + b]$ , which by equation (21) falls within the linear range above the inflation target,  $(\pi^* + c, \infty)$ .

The marginal loss function, shown in Figure 4, consists of the segment DE, BB', and E'D'. This loss function can be seen as a simplification of the “bell loss function,”

$$L(\pi) = c \left( 1 - e^{-(\pi - \pi^*)^2/2} \right),$$

which is discussed and examined by Bray and Goodhart (2002). The bell loss function is close to a quadratic loss function for small deviations from the target and asymptotically approaches an upper bound  $c$  for large deviations. The marginal loss of the bell loss function is continuous,

$$\frac{\partial L(\pi)}{\partial \pi} = c(\pi - \pi^*) e^{-(\pi - \pi^*)^2/2},$$

approximately linear and increasing for small deviations from the inflation target and approaching zero for large deviations. It can be motivated by a finite upper bound for the loss, for instance, if the worst thing that can happen to a central banker is to be fired from the job after bad performance and this is associated with a finite loss.

The expected marginal loss of the combined quadratic/constant loss function is then

$$E \frac{\partial L(\pi)}{\partial \pi} = \int_{\pi^* - c}^{\pi^* + c} (\pi - \pi^*) h(\pi - \bar{\pi}) d\pi = E[\pi - \pi^*; |\pi - \pi^*| \leq c] \Pr[|\pi - \pi^*| \leq c].$$

Setting this equal to zero, according to the optimal targeting rule, results in

$$E[\pi; |\pi - \pi^*| \leq c] = \pi^*. \quad (24)$$

Expected inflation conditional on inflation falling within the quadratic range of the loss function should equal the inflation target.

In the special case of inequality (21), we have  $E[\pi; |\pi - \pi^*| \leq c] = \bar{\pi}$ . It follows that optimal normal mean inflation under a combined quadratic/constant loss function, denoted  $\bar{\pi}^{qc}$ , fulfills

$$\bar{\pi}^{qc} = \pi^*.$$

The unconditional mean inflation then fulfills

$$E\pi = \pi^* + \gamma a > \pi^*.$$



## 6 A Perfectionist Loss Function: Mode Inflation Targeting

Finally, let us consider a perfectionist, all-or-nothing, loss function, a rather special case where only an outcome right on the inflation target is acceptable and everything else is a failure. This can be represented with the help of Dirac's Delta function,  $\delta(x)$ , which has the properties

$$\delta(x) = \begin{cases} \infty & \text{for } x = 0, \\ 0 & \text{for } x \neq 0, \end{cases}$$

and

$$\int \delta(x)f(x)dx = f(0)$$

for any function  $f(x)$ . The perfectionist loss function can then be written

$$L(\pi) \equiv -\delta(\pi - \pi^*),$$

giving an unbounded gain (negative loss) when inflation equals the inflation target and zero otherwise. The Delta function can be seen as the limit of a sequence of  $\delta_n(x)$ -functions, the tent-shaped function illustrated in Figure 6, when  $n$  approaches infinity,

$$\delta(\pi - \pi^*) = \lim_{n \rightarrow \infty} \delta_n(\pi - \pi^*).$$

I avoid illustrating the Delta function in Figure 3. It would be an infinitesimally thin negative spike extending to minus infinity located at  $\pi = \pi^*$  and equal to zero elsewhere.

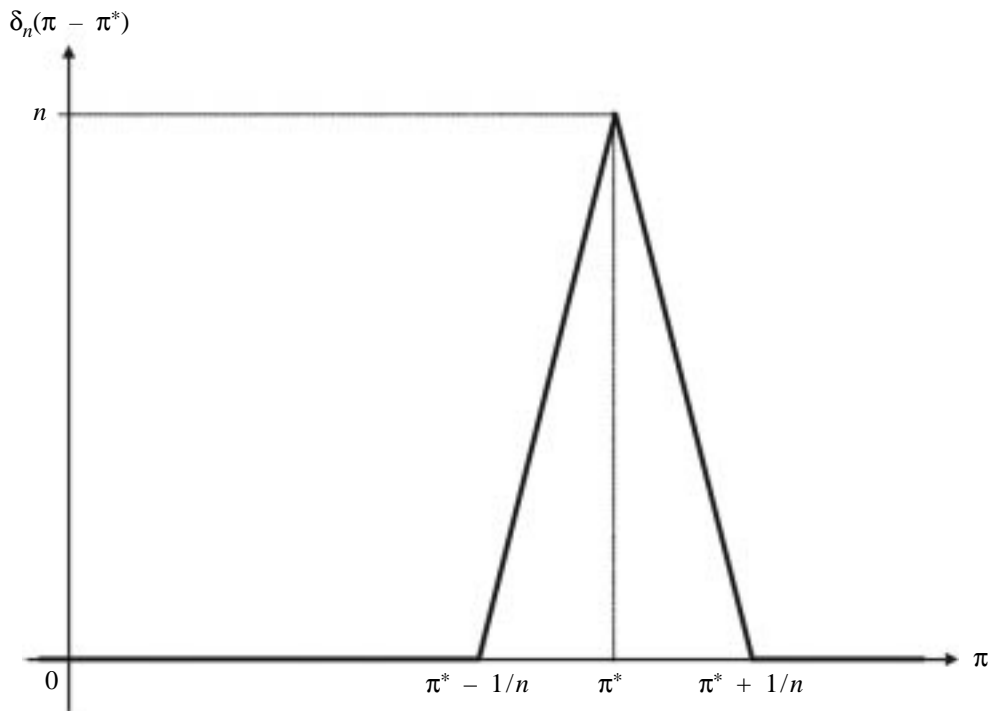
The Delta function is not differentiable at  $\pi = \pi^*$ , so we cannot specify the marginal loss in this case. Therefore, the optimal targeting rule stating that the expected marginal loss should equal zero does not apply here. Instead, we have to look directly at the expected loss. It is

$$EL(\pi) \equiv \int L(\pi)h(\pi - \bar{\pi})d\pi = -h(\pi^* - \bar{\pi}),$$

the negative of the density function for inflation at the inflation target. Minimizing the loss then corresponds to adjusting normal mean inflation,  $\bar{\pi}$ , so as to maximize the probability density of an inflation outcome equal to the inflation target. This means adjusting normal mean inflation such that the mode of inflation coincides with the inflation target, *mode inflation targeting*,

$$\text{mode}(\pi) = \pi^*.$$

**Figure 6**  
**The  $\delta_n(\pi - \pi^*)$  function**



It follows from equation (4) that the optimal normal mean inflation under the perfectionist loss function,  $\bar{\pi}^P$ , is given by

$$\bar{\pi}^P = \pi^* - \text{mode}(z).$$

For the density function  $h(z)$  in equation (1), since the probability of the extreme event,  $\gamma$ , is a small number, the mode of  $h(z)$  coincides with the mode of  $f(z)$ , the distribution for the normal shock,

$$\text{mode}(z) = \text{mode}(z; \eta = 0).$$

That is, since the extreme event has low probability, the mode occurs for the normal outcome and is independent of the probability and the size of the extreme event. Thus, mode inflation targeting completely disregards the extreme event (as long as it has low probability).

For a symmetric unimodal distribution for the normal outcome, the mode, the mean, and the median are all equal. This is the case for the probability distribution  $f(z)$  shown in Figure 1, for which  $\text{mode}(z)$  is zero. Hence, in this case, optimal normal inflation under the perfectionist loss function,  $\bar{\pi}^P$ , equals the inflation target,

$$\bar{\pi}^P = \pi^*.$$

For distributions that are not symmetric or unimodal, the mode, the mean, and the median will generally differ from each other. Furthermore, the mode is not necessarily unique. The uniform probability distribution, assumed in the special case and shown in Figure 5, has the mode equal to the whole support of the distribution for the normal outcome,

$$-b \leq \text{mode}(z) \leq b.$$

Thus, in this case, the optimal normal mean inflation under the perfectionist loss function, denoted  $\bar{\pi}^P$ , is not unique but given by the interval

$$\pi^* - b \leq \bar{\pi}^P \leq \pi^* + b,$$

which we can write as  $\bar{\pi}^P = \pi^* + [-b, b]$ .

Thus, a rather bizarre loss function is required to give prominence to mode projections. Furthermore, mode projections have some undesirable properties. Of possible measures of the “central tendency” of a projection, it seems less relevant than the mean or the median. Some central banks nevertheless present their main projections as mode projections. This practice seems difficult to defend, as noted by Svensson (2001), for instance.

## 7 An Endogenous Extreme Shock

Above, the size and probability of the extreme shock have been taken as exogenous. A possible relevant case, however, is when the size and/or the probability of the extreme shock depends on the policy instrument. For instance, the bursting of an asset-price bubble may depend on the policy.<sup>7</sup>

A simple case is when the size of the extreme shock,  $a$ , is a linear function of the instrument,

$$a = a_1 + a_2 i,$$

where  $a_1$  is a given large positive number and  $a_2$  is a given number. Let us assume that the coefficient  $a_2$  is positive, that is, the size of the extreme inflationary shock is increasing in the instrument. This would seem to fit the U.K. exchange rate example, since a higher interest rate might increase the potential overvaluation of the pound and thereby the later depreciation.

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7. Pierre Duguay and Charles Goodhart made this point at the conference, which led me to add this section to the paper. The section has benefited from additional comments from Charles Goodhart.

Generally, the sign of the coefficient  $a_2$  could depend on the nature of the shock. The case above with exogenous  $a$  obviously corresponds to  $a_2 = 0$ . We assume that  $a$  remains a large positive number for the relevant range of the instrument.

The probability density function of the shock  $z$  can then be written as

$$h(z, i) \equiv (1 - \gamma)f(z) + \gamma f(z - a_1 - a_2 i),$$

where the instrument  $i$  now has a special direct effect on the density function, since  $a_2$  is different from zero. It follows that the density function of inflation can be written

$$h(\pi - \bar{\pi}, i) = (1 - \gamma)f(\pi - \bar{\pi}) + \gamma f(\pi - \bar{\pi} - a_1 - a_2 i),$$

again with a direct effect of the instrument on the density, in addition to the standard effect on normal mean inflation in equation (3).

The optimal policy now has to take the special direct effect of the instrument into account. The expected loss is

$$EL(\pi) \equiv \int L(\pi)h(\pi - \bar{\pi}, i)d\pi = \int L(\bar{\pi} + z)h(z, i)dz.$$

The derivative of the expected loss with respect to the instrument is

$$\frac{\partial}{\partial i}EL(\pi) = \int \frac{\partial L(\bar{\pi} + z)}{\partial \pi} \frac{\partial \bar{\pi}}{\partial i} h(z, i)dz + \int L(\bar{\pi} + z) \frac{\partial h(z, i)}{\partial i} dz.$$

Since  $\frac{\partial \bar{\pi}}{\partial i} = -\alpha$  by equation (3), the first term on the right side can be written

$$\int \frac{\partial L(\bar{\pi} + z)}{\partial \pi} \frac{\partial \bar{\pi}}{\partial i} h(z, i)dz = E \frac{\partial L(\pi)}{\partial \pi} (-\alpha),$$

the expected marginal loss times  $-\alpha$ . The second term can be written

$$\begin{aligned} \int L(\bar{\pi} + z) \frac{\partial h(z, i)}{\partial i} dz &= \Pr[\eta = a] \frac{\partial}{\partial i} \int L(\bar{\pi} + a_1 + a_2 i + z) f(z) dz \\ &= \gamma \int \frac{\partial L(\bar{\pi} + a_1 + a_2 i + z)}{\partial \pi} f(z) dz a_2 \\ &= \gamma E \left[ \frac{\partial L(\pi)}{\partial \pi}; \eta = a \right] a_2, \end{aligned}$$

the probability of the extreme event times the expected loss conditional on the extreme event times  $a_2$ .

The first-order condition with respect to the instrument sets the derivative

$$\frac{\partial}{\partial i} EL(\pi) = 0.$$

This, together with the above expressions, results in the optimal targeting rule,

$$E \frac{\partial L(\pi)}{\partial \pi} (-\alpha) + \gamma E \left[ \frac{\partial L(\pi)}{\partial \pi}; \eta = a \right] a_2 = 0. \quad (25)$$

It can be interpreted as follows. A marginal increase  $di > 0$  of the instrument shifts the whole probability distribution of inflation in Figure 2 to the left by  $\alpha di$ . The corresponding marginal change of the expected loss is

$$E \frac{\partial L(\pi)}{\partial \pi} (-\alpha) di.$$

Furthermore, this increases the size of the extreme shock by  $da = \alpha_2 di$  and shifts the extreme part of the probability distribution of inflation in Figure 2, the density function around  $\bar{\pi} + a$ , to the right by the same amount. The corresponding marginal change of the expected loss is

$$\gamma E \left[ \frac{\partial L(\pi)}{\partial \pi}; \eta = a \right] a_2 di,$$

the probability of the extreme shock times the expected marginal loss conditional on the extreme event times the increase in the size of the extreme shock. The sum of these marginal changes in the expected loss must be zero in an optimum, which gives equation (25).

Policy is affected by the endogeneity of the size of the shock only when the marginal loss conditional on the extreme shock is non-zero. Then we will have

$$E \frac{\partial L(\pi)}{\partial \pi} > 0 \text{ instead of } E \frac{\partial L(\pi)}{\partial \pi} = 0,$$

and optimal normal mean inflation will be higher than when the size of the extreme shock is exogenous. It will be optimal to lower the size of the extreme shock somewhat, but this requires a lower instrument level and results in a higher normal mean inflation.

For the *quadratic* loss function, we have

$$E \frac{\partial L(\pi)}{\partial \pi} = E[\pi - \pi^*] = \bar{\pi}^q + \gamma a^q - \pi^*, \quad (26)$$

$$\gamma E \left[ \frac{\partial L(\pi)}{\partial \pi}; \eta = a \right] = \gamma E[\pi - \pi^*; \eta = a] = \gamma(a^q - \pi^*), \quad (27)$$

where  $\bar{\pi}^q$  and  $a^q$  denote the optimal normal mean inflation and size of the extreme shock, respectively. They are related by

$$\bar{\pi}^q = x - \alpha i^q, \quad (28)$$

$$a^q = a_1 + a_2 i^q, \quad (29)$$

where  $i^q$  denotes the optimal instrument setting. Substitution of equations (26) and (27) into equation (25) leads to the optimal targeting rule

$$\bar{\pi}^q = \pi^* - \gamma a^q + \gamma(a^q - \pi^*) \frac{a_2}{\alpha}. \quad (30)$$

Substitution of equations (28) and (29) into equation (30) allows us to solve for the optimal  $i^q$  and thereby  $\bar{\pi}^q$  and  $a^q$ . Since we have  $a^q > \pi^*$  (recall that  $a$  is assumed to remain a large positive number and  $\pi^*$  is a low positive number), we see from equation (30) that normal mean inflation will undershoot the inflation target by less than the mean size of the extreme shock. If  $a_2/\alpha > a^q/(a^q - \pi^*)$ , normal mean inflation will even overshoot rather than undershoot the inflation target.

For the *combined quadratic/absolute deviation* loss function, we have

$$E \frac{\partial L(\pi)}{\partial \pi} = (1 - \gamma)E[\pi - \pi^*; \eta = 0] + \gamma c = (1 - \gamma)(\bar{\pi}^{qa} - \pi^*) + \gamma c,$$

$$\gamma E \left[ \frac{\partial L(\pi)}{\partial \pi}; \eta = a \right] = \gamma c,$$

the latter under the assumption that the support of inflation for the extreme shock falls within the linear range. Substituting this into equation (25) gives

$$\bar{\pi}^{qa} = \pi^* - \frac{\gamma}{1 - \gamma} c \left( 1 - \frac{a_2}{\alpha} \right).$$

The optimal normal mean inflation undershoots the inflation target by less than when the size of the extreme shock is exogenous, cf. equation (22). If  $a_2 > \alpha$ , the optimal normal mean inflation even overshoots the inflation target.

For the *absolute-deviation* loss function, we have

$$E \frac{\partial L(\pi)}{\partial \pi} = 1 - 2\Pr[\pi \leq \pi^*],$$

$$\gamma E \left[ \frac{\partial L(\pi)}{\partial \pi}; \left| \eta = a \right. \right] = \gamma.$$

Substituting this into equation (25) gives

$$\Pr[\pi \leq \pi^*] = \frac{1}{2} \left( 1 - \gamma \frac{a_2}{\alpha} \right), \quad (31)$$

which implies that the median inflation will overshoot the inflation target. Let  $\tilde{z}$  be defined by

$$\Pr[z \leq \tilde{z}] \equiv \frac{1}{2} \left( 1 - \gamma \frac{a_2}{\alpha} \right).$$

We have  $\tilde{z} < \text{med}(z)$ . Then,

$$\bar{\pi}^a = \pi^* - \tilde{z} > \pi^* - \text{med}(z).$$

The optimal normal mean inflation undershoots the inflation target by less than when the size of the extreme shock is exogenous.

Under the special case of a uniform distribution, equation (15), we have

$$\Pr[z \leq \tilde{z}] = (1 - \gamma) \frac{1}{2b} (\tilde{z} + b) \quad \text{for } -b \leq z \leq b.$$

Combining this with equation (31) gives

$$\tilde{z} = \frac{\gamma}{1 - \gamma} b \left( 1 - \frac{a_2}{\alpha} \right) < \text{med}(z) = \frac{\gamma}{1 - \gamma} b.$$

For  $a_2 > \alpha$ , we have  $\tilde{z} < 0$ , and optimal mean inflation will overshoot the inflation target.

For the *combined quadratic/constant* loss function and the *perfectionist* loss function, the endogeneity of the size of the extreme shock does not affect the optimal policy, since then  $E[\partial L(\pi)/\partial \pi; \eta = a] = 0$ .

## 8 Comparing Optimal Policy for the Different Loss Functions

Comparing the resulting optimal normal mean inflation for the different loss functions when the size of the extreme shock is exogenous, we have

$$\bar{\pi}^q = \pi^* - \text{mean}(z) < \bar{\pi}^{qa} = \pi^* - \frac{\gamma}{1-\gamma}c < \bar{\pi}^a = \pi^* - \text{med}(z) < \bar{\pi}^{qc} = \pi^* \underset{>}{\leq} \bar{\pi}^p = \pi^* - \text{mode}(z),$$

where  $\text{mean}(z) = \gamma a$ , and where the median and the mode of  $z$  depend on the distribution  $f(z)$ . In the special case of inequalities (13) and (21) and equation (14), we have

$$\bar{\pi}^q = \pi^* - \gamma a < \bar{\pi}^{qa} = \pi^* - \frac{\gamma}{1-\gamma}c < \bar{\pi}^a = \pi^* - \frac{\gamma}{1-\gamma}b < \bar{\pi}^{qc} = \pi^* \underset{>}{\leq} \bar{\pi}^p = \pi^* + [-b, b].$$

The quadratic loss function implies setting unconditional mean inflation equal to the inflation target and hence letting mean inflation conditional on the normal outcome, normal mean inflation,  $\bar{\pi}^q$ , undershoot the target by the mean size of the extreme event,  $\gamma a$ , the probability-weighted *size* of the extreme event. Thus, the amount of undershooting depends both on the probability,  $\gamma$ , and the size,  $a$ , of the extreme event. The quadratic/absolute-deviation loss function implies letting normal mean inflation,  $\bar{\pi}^{qa}$ , undershoot the inflation target by the probability-weighted *constant marginal loss* for the extreme event,  $\gamma c$ , which is less than the probability-weighted size of the extreme event. Once the extreme shock is so large that inflation then falls in the linear range, the weight on the extreme event is independent of the size of the extreme event.

The absolute-deviation loss function implies setting median inflation equal to the target; hence, letting the normal mean inflation,  $\bar{\pi}^a$ , undershoot the target by the median of the total shock to inflation,

$$\text{med}(z) = \frac{\gamma}{1-\gamma}b.$$

This puts even less weight on the extreme event (by inequality (21),  $b$  is less than  $c$ ). Again, the monetary policy response is independent of the size of the extreme event. Finally, the quadratic/constant loss function completely ignores the extreme event, letting the normal mean inflation,  $\bar{\pi}^{qc}$ , equal the



inflation target. It ignores the extreme event when the shock is large enough to fall in the constant range of the loss function, regardless of its probability.

The perfectionist loss function leads to a focus on mode projections and thereby ignores the extreme event because it is low-probability, regardless of its size. Whether this implies that normal mean inflation under- or overshoots the target depends on whether or not the mode of the normal shock under- or overshoots the mean.

We have also considered the case when the size of the extreme shock is endogenous and, more precisely, a linear function of the policy instrument. We noted that whether the size of the extreme shock is increasing or decreasing depends on the nature of the shock, and we took as the main case the one where the size is increasing in the instrument, which fits the U.K. exchange rate example. Then the size of the extreme shock and normal mean inflation move in opposite directions when the instrument is adjusted. Optimal normal mean inflation will then undershoot the inflation target less for the quadratic, combined quadratic/absolute-deviation and absolute-deviation loss functions than when the size of the extreme shock is exogenous. If the effect on the size of the extreme shock of the instrument is sufficiently large relative to the effect on inflation, the optimal normal mean inflation will even overshoot the inflation target in those cases. The reason is that lowering the instrument and increasing the normal mean inflation will reduce the size and thereby the expected loss from the extreme shock. For the combined quadratic/constant and perfectionist loss functions, endogeneity of the size of the extreme shock has no effect on policy, since the expected loss from the extreme shock is then independent of its size.

Which of these different loss functions makes most sense for monetary policy? This is, of course, a rather deep question. The answer depends on, for instance, whether the loss function is an approximation to social welfare, assigned to the central bank by society or political authorities as in a principal-agent situation, an interpretation by the central bank of less precise instructions or legislation, or the private loss function of a decision-maker. In the latter case, Bray and Goodhart (2002) argue that there is nothing worse than being fired, for which situation they find their bell loss function with an upper bound appropriate. However, even in this case, a decision-maker can be fired with more or less personal disgrace, making the fixed upper bound somewhat doubtful.

One way to evaluate the sensibility of the loss functions is to consider the marginal loss. Does it make sense that the marginal loss is increasing, constant, or decreasing in the deviation from the target? Is the marginal loss increasing for small deviations and constant or decreasing for large ones?

I hope I am excused for not having any definite answers to these questions, except that the perfectionist loss function seems rather extreme.

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