Liquidity and Bank Capital Requirements

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Abstract

A dynamic competitive equilibrium model in this paper incorporates illiquidity of assets due to asymmetric information about asset quality. In the model, both a negative productivity shock and an increase in the degree of asymmetric information can cause a simultaneous deterioration of illiquidity of assets and the market price of assets. Illiquidity of assets leads to liquidity transformation by banks, and banks finance part of their assets through public equities (bank capital) to prevent a bank run in equilibrium. The capital-asset ratio of banks increases in illiquidity of bank assets and the volatility of the market price of bank assets.

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1 Introduction

This paper presents a dynamic competitive equilibrium model in which illiquidity of assets arises due to asymmetric information about asset quality. There are two main results. First, it is shown that both a negative productivity shock and a rise in the degree of asymmetric information can cause an increase in illiquidity of assets and a drop in the market price of assets, as occurred during the financial crisis since 2007. Second, the model shows that illiquidity of assets leads to financial intermediation for liquidity transformation. It is found that liquidity-transforming banks are necessarily susceptible to a self-fulfilling bank run due to illiquidity of bank assets and need to finance part of their assets through public equities (bank capital) to prevent a bank run. The equilibrium capital-asset ratio of banks increases in illiquidity of bank assets and the volatility of the market price of bank assets.

The model is a version of the AK model, where only a part of agents can produce new capital from goods due to an idiosyncratic shock to each agent. Call the agents who can produce new capital as 'productive' and those who cannot as 'unproductive'. Agents are anonymous and cannot borrow against future income. In the model, the number of the productive is so small that the amount of goods produced from the productive's own capital is short of the efficient level of aggregate investment in new capital. The productive sell their used capital to obtain goods from the unproductive for maximizing their investments in new capital.

The competitive market for capital, however, is contaminated by adverse selection, since every unit of used capital depreciates at its own rate and the depreciation rate is private information for the holder of the unit of used capital. The adverse selection lowers the market price of capital, leading to a market's undervaluation of the average quality of used capital held by each agent. Thus, used capital held by each agent is illiquid as a whole. In this paper, illiquidity is defined as undervaluation in the market.

The degree of illiquidity of capital fluctuates in response to shocks. There are two types of shocks in the model; a productivity shock and a change in the range of possible depreciation rates of capital. It is shown that both types of shocks can cause an increase in illiquidity of capital and a decline in the market price of capital, as occurred during the financial crisis since 2007. First, a negative productivity shock reduces the market price of capital, since a decline in agents' income lowers aggregate spending on capital. A decline in the market price of capital in turn discourages agents from selling high-quality capital, which increases illiquidity of capital due to worsened adverse selection. Second, an expansion of the range of possible depreciation rates of capital increases the degree of asymmetric information in the market for capital. This effect of the shock worsens adverse selection, and a resulting increase in illiquidity of capital lowers the market price of capital.

Illiquidity of capital also explains why financial intermediation is necessary for the economy. Banks can supply liquid securities to agents by holding illiquid capital and financing the cost through bank deposits. Since idiosyncratic depreciation rates of capital held by banks cancel each other out, the average quality of each bank's assets becomes public information, which makes bank deposits issued against bank assets free from illiquidity. Agents can increase investments in new capital by storing wealth through bank deposits when they are unproductive and selling them when they are productive.

It is found that liquidity-transforming banks would be susceptible to a self-fulfilling bank run if they issued bank deposits up to the true value of bank assets, since repayable bank deposits would exceed the liquidation value of bank assets due to illiquidity of capital.[1] To prevent a bank run, banks need to finance the difference between the true value and the liquidation value of their assets through public equities (bank capital).

The dynamic analysis of the model identifies two factors that determine the minimum capital-asset ratio for banks to prevent a bank run. First, to eliminate the possibility of a

[1]This is the same type of the panic-based bank run as analyzed by Diamond and Dybvig (1983).

bank run, banks must limit the repayable amount of bank deposits to the liquidation value of their assets in the next possible recession. This factor drives the minimum capital-asset ratio of banks to be pro-cyclical, since the true value of bank assets increases during booms, while the limit on bank deposits remains equal to the liquidation value of bank assets during recessions. Second, when negative shocks hit the economy, illiquidity of capital increases due to worsened adverse selection, as explained above. This factor enlarges the difference between the true value and the liquidation value of bank assets, which drives the minimum capital-asset ratio of banks to be counter-cyclical. Overall, the capital-asset ratio of banks increases in illiquidity of bank assets and the volatility of the market price of bank assets, and the relative balance between these two factors determines the cyclicality of the capital-asset ratio of banks. In the numerical examples of the dynamics of the model, the capital-asset ratio of banks is pro-cyclical when business cycles are driven by productivity shocks and it is counter-cyclical when business cycles are driven by changes in the degree of asymmetric information (i.e., changes in the range of possible depreciation rates of capital).

The most related paper to this paper is Kiyotaki and Moore (2005). They introduce a resaleability constraint on capital in a dynamic competitive equilibrium model. This constraint is interpreted as controlling for the effectiveness of financial intermediation that alleviates asymmetric information by bunching assets with idiosyncratic qualities. This paper formalizes this interpretation and endogenizes the dynamics of liquidity of assets and bank capital requirements. Also, this paper contributes to the literature on dynamic competitive equilibrium models of banking, in which banks are usually either entrepreneurs whose bank capital is their internal net-worth or firms that earn zero profit and do not need to maintain any bank capital.[2] The model in this paper incorporates publicly-owned banks subject to bank capital requirements, which arise from asymmetric information endogenously.

[2]For example, see Williamson (1987), Bernanke and Gertler (1989), Holmström and Tirole (1997), and Chen (2001).

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 derives aggregate equilibrium conditions. Section 4 analytically solves the model without a banking sector. Section 5 analyzes liquidity transformation by banks in equilibrium and the dynamics of the capital-asset ratio of banks. Section 6 concludes.

2 The model

2.1 The agent's problem

There is a discrete-time economy with a continuum of infinite-lived agents and a representative bank in a competitive banking sector. Denote the set of agents in the economy by I and the measure on the continuum of agents by μ.

Each agent can produce homogeneous goods from capital in the beginning of every period. The production function for goods is:

$$y_{i,t} = \alpha_t k_{i,t-1}, \quad \alpha_t \in \{\bar{\alpha}, \underline{\alpha}\}, \tag{1}$$

where i is an index for each agent, t denotes a time period, $y_{i,t}$ is output, $k_{i,t-1}$ is capital held at the end of the previous period, and α_t is the productivity of capital common to all the agents.

Capital is divisible and each infinitesimal unit of capital depreciates at its own rate after production. Denote by $k_{i,\delta,t-1}$ the density of capital that is used by agent i and depreciates at the rate of δ after the production in period t. Depreciation rates are independently and identically distributed by a uniform distribution, such that:

$$\delta \sim U[\bar{\delta} - \Delta_t,\ \bar{\delta} + \Delta_t], \quad \Delta_t \in \{\bar{\Delta}, \underline{\Delta}\}, \tag{2}$$

$$k_{i,\delta,t-1} = \frac{k_{i,t-1}}{2\Delta_t}, \tag{3}$$

where $\bar{\delta} \in (0,1)$. Note that $(2\Delta_t)^{-1}$ is the density of the uniform distribution.

Each value of α_t and Δ_t is determined by a Markov process. For $x = \alpha, \Delta$ and for all t, the conditional probability that $x_{t+1} = x_t = \bar{x}$ is $\bar{\eta}_x$ ($\in [0,1]$) and the conditional probability that $x_{t+1} = x_t = \underline{x}$ is $\underline{\eta}_x$ ($\in [0,1]$). Assume $\bar{\alpha}, \underline{\alpha} > 0$ and that $\bar{\Delta}, \underline{\Delta} \in (0, 1-\bar{\delta})$.

Assume that the depreciation rate of each infinitesimal unit of capital is private information for the agent who uses the unit of capital for production in the beginning of each period. The depreciation rate becomes public information when the capital is used for production again in the next period, which reveals the depreciation rate through the amount of goods produced by the capital.[3]

Call depreciated capital after production as 'used capital'. Agents can trade used capital in a competitive market, where a price is set to each infinitesimal unit of capital. Assume that agents are anonymous and that the price of used capital in the market cannot be contingent on the characteristics of the buyer or the seller. As a consequence, every unit of capital is traded at an identical price in each period.[4] By the law of large numbers, the realized average depreciation rate of used capital bought by each buyer equals the average depreciation rate of used capital sold in the market, which is denoted by $\hat{\delta}_t$.[5]

Only a part of agents can produce new capital from goods. The production function for new capital is:

$$i_{i,t} = \phi_{i,t} x_{i,t}, \quad \phi_{i,t} \in \{0, \phi\}, \tag{4}$$

[3]After the revelation of depreciation rates, capital net depreciation becomes homogeneous once and then each unit of capital depreciates at its own rate.

[4]If there are multiple competitive markets sorted by the amount of capital sold by each seller, then the quantity of sold capital could signal the quality of the capital. Even in this case, anonymity of sellers would let each seller split her sold capital in multiple lots and sell them in different markets to maximize the total revenue from the sales. This paper abstracts from the interaction between competitive market prices and this type of seller's strategic behaviour.

[5]This is a common feature of competitive equilibrium models with adverse selection. See Gale (1992) and Eisfeldt (2004) for example.

where $i_{i,t}$ is newly produced capital and $x_{i,t}$ is the amount of goods invested in capital. The productivity of investment, $\phi_{i,t}$, is determined by an idiosyncratic Markov process for each agent. The conditional probability that $\phi_{i,t+1} = \phi_{i,t} = \phi$ is ρ_P ($\in [0,1]$) and the conditional probability that $\phi_{i,t+1} = \phi_{i,t} = 0$ is ρ_U ($\in [0,1]$) for all i and t. The new capital only materializes in the beginning of the next period before the timing of production and cannot be traded today.

Each agent maximizes expected utility from consumption every period. The agent's maximization problem is:

$$\max_{\{c_{i,s},\, x_{i,s},\, k^o_{i,s},\, l_{i,\delta,s}\}_{s=0}^{\infty}} E_t \sum_{s=0}^{\infty} \beta^s \ln c_{i,s}$$

$$\begin{aligned} \text{s.t. } & c_{i,s} + x_{i,s} + Q_s k^o_{i,s} + b_{i,s} + (1+\zeta) V_s s_{i,s} \\ & = \alpha_s k_{i,s-1} + Q_s \int_{\bar{\delta}-\Delta_s}^{\bar{\delta}+\Delta_s} l_{i,\delta,s}\, d\delta + R_s b_{i,s-1} + (D_s + V_s) s_{i,s-1}, \\ & k_{i,s} = \phi_{i,s} x_{i,s} + (1-\hat{\delta}_s) k^o_{i,s} + \int_{\bar{\delta}-\Delta_s}^{\bar{\delta}+\Delta_s} (1-\delta)\left(k_{i,\delta,s-1} - l_{i,\delta,s}\right)\, d\delta, \\ & l_{i,\delta,s} \in [0,\ k_{i,\delta,s-1}], \\ & c_{i,s},\ x_{i,s},\ k^o_{i,s}\ b_{i,s},\ s_{i,s} \geq 0, \end{aligned} \tag{5}$$

where $\beta \in (0,1)$ and $\zeta > 0$.

The first constraint of the maximization problem (5) is a flow-of-fund constraint, where $c_{i,s}$ is consumption, $k^o_{i,t}$ is the amount of used capital bought from the market, $l_{i,\delta,t}$ is the density of used capital with a depreciation rate δ sold by the agent in the market, $b_{i,s}$ is the amount of one-period bank deposits, $s_{i,s}$ is the number of bank equities, Q_s is the market price of used capital, R_s is the ex-post deposit interest rate, V_s is the ex-dividend price of bank equities, D_s is the amount of bank dividends per equity, and ζ is a marginal cost of holding bank equities. The bank-equity holding cost is a reduced-form representation of costs of managing equities, such as a transaction cost and a cost of monitoring that is necessary to make the bank to pay dividends.[6] The existence of this cost will make agents require a higher rate of returns on bank equities than bank deposits. Thus, equity financing becomes costly for the bank. The flow-of-fund constraint also incorporates an assumption that agents cannot borrow against their future income due to their anonymity that makes it difficult to enforce their commitments.

The second constraint is the law of motion of capital. The third constraint means that the sales of used capital must be non-negative and that the agent cannot sell more than the amount of used capital the agent owns. The fourth constraint is a non-negativity constraint on choice variables. Each agent takes as given the probability distribution of $\{Q_s, D_s, V_s, R_s, \hat{\delta}_s, \alpha_s, \Delta_s, \phi_{i,s}\}_{s=0}^{\infty}$.

2.2 The bank's problem

The representative bank in the competitive banking sector can buy used capital from the market and sell bank equities and one-period bank deposits to agents. The implicit assumption behind the difference in the ability to borrow (including equity financing) between agents and the bank is that the bank is not anonymous, which makes it feasible to enforce its commitments. In contrast, agents are anonymous and it is difficult to enforce their commitments. Assume that writing contingent contracts that are enforceable by the court is still costly, so that the bank can only issue deposits and equities.[7]

If there are equity holders of the bank, then the bank maximizes the value of the bank for equity holders. In this case, the bank maximizes the value of $(D_t + V_t)s_{i,t-1}$ for each agent every period, since the maximum of each agent's utility function increases in the agent's wealth, given the probability distribution of exogenous variables for agents. Since

[6]The predetermined amounts of repayments to deposits are protected by laws. In contrast, equity holders need to find the total cash flows for the bank and negotiate with the bank on the amount of dividends.

[7]Note that equities are not contingent contracts that specify contingent returns ex-ante. Instead, ex-post negotiation on dividends must take place as if default on debt occurs every period. See Hart and Moore (1994) for more details on the feature of equities as a financial contract.

$\{s_{i,t-1}|i \in I\}$ is predetermined, maximizing $(D_t + V_t)s_{i,t-1}$ for all $i \in I$ is equivalent to maximizing the total value of the bank for equity holders, $(D_t+V_t)S_{B,t-1}$, where $S_{B,t}$ denotes the number of bank equities issued by the bank. The total value of the bank for equity holders is derived from a flow-of-fund constraint on the bank:

$$D_t S_{B,t-1} + R_t B_{B,t-1} + Q_t(K^o_{B,t} - L_{B,t}) = \alpha_t K_{B,t-1} + B_{B,t} + V_t(S_{B,t} - S_{B,t-1}), \qquad (6)$$

where $K_{B,t-1}$ is the amount of capital held at the end of the previous period, $L_{B,t}$ is the amount of used capital sold by the bank, $K^o_{B,t}$ is the amount of used capital bought from the market, and $B_{B,t}$ is the amount of bank deposits issued by the bank. Note that the last term on the right-hand side of the equation is the revenue from newly issued equities or the expenditure on equity repurchase.

If the bank fulfills deposit contracts, then the ex-post deposit interest rate, R_t, is the ex-ante non-contingent interest rate specified by deposit contracts in the previous period, which is denoted by $\tilde{R}_{t-1}$. If the bank defaults, then the ex-post deposit interest rate equals the recovery rate of deposits. Assume that a bank run occurs if the repayable amount of deposits exceeds the liquidation value of capital held by the bank.[8] In this case, the bank cannot roll over its deposits and must maximize the repayment to depositors by liquidating all the capital it owns. Since the liquidation value of the bank's capital is less than the

[8]As shown below, the present discounted value of future income generated by the bank's capital exceeds the liquidation value of the capital. Thus, if the bank can roll over deposits, the bank can avoid default. But if all the depositors expect that the bank cannot roll over deposits in this case, then their expectations are self-fulfilling.

repayable amount of deposits, the bank must default and bank equities lose value. Thus:

$$\begin{cases} B_{B,t}, D_t, V_t, K^o_{B,t} = 0,\ L_{B,t} = K_{B,t-1},\ R_t = \frac{(\alpha_t + Q_t) K_{B,t-1}}{B_{B,t-1}} & \text{if } \tilde{R}_{t-1} B_{B,t-1} > (\alpha_t + Q_t) K_{B,t-1}, \\ R_t = \tilde{R}_{t-1} & \text{if } \tilde{R}_{t-1} B_{B,t-1} \leq (\alpha_t + Q_t) K_{B,t-1}. \end{cases} \tag{7}$$

Note that the recovery rate of deposits, R_t, on the first line is determined by the flow-of-fund constraint (6).

When maximizing $(D_t+V_t)S_{B,t-1}$, the bank internalizes the price of bank equities, V_t, and the ex-ante deposit interest rate, $\tilde{R}_t$, the latter of which responds to the level of bank deposits through the probability of a bank run in the next period. These prices are determined by the first-order conditions for bank securities in the agent's maximization problem (5):

$$(1+\zeta)V_t \geq E_t\left[\frac{\beta c_{i,t}(D_{t+1} + V_{t+1})}{c_{i,t+1}}\right], \tag{8}$$

$$1 \geq E_t\left[\frac{\beta c_{i,t} R_{t+1}}{c_{i,t+1}}\right]. \tag{9}$$

The first line is for bank equities and the second line for bank deposits. On each line, the left-hand side of the weak inequality is the marginal cost of the bank security in terms of current consumption and the right-hand side is the marginal return from the bank security. If the strict inequality holds, then the agent does not hold the bank security. The equality holds for the agent who has the largest value of the right-hand side term among agents. Denote the indices of these agents for bank equities and bank deposits by $i_{S,t}$ and $i_{B,t}$, respectively.

The total value of the bank is determined by a recursive maximization problem such that:

$$
\begin{aligned}
&(D_t + V_t)S_{B,t-1} = \Omega_t(K_{B,t-1}, B_{B,t-1}, \tilde{R}_{t-1}) \equiv \\
&\max_{\{K^o_{B,t}, L_{B,t}, B_{B,t}, \tilde{R}_t\}} \quad \alpha_t K_{B,t-1} + Q_t(L_{B,t} - K^o_{B,t}) - R_t B_{B,t-1} + B_{B,t} \\
&\qquad\qquad + E_t\left[\left. \frac{\beta c_{i,t} \Omega_{t+1}(K_{B,t}, B_{B,t}, \tilde{R}_t)}{(1+\zeta) c_{i,t+1}} \right| i = i_{S,t} \right], \\
&\qquad \text{s.t. } 1 = E_t\left[\left. \frac{\beta c_{i,t} \min\left\{\tilde{R}_t,\ (\alpha_{t+1} + Q_{t+1}) K_{B,t} (B_{B,t})^{-1}\right\}}{c_{i,t+1}} \right| i = i_{B,t} \right], \qquad (10) \\
&\qquad K_{B,t} = (1 - \hat{\delta}_t) K^o_{B,t} + (1 - \bar{\delta})(K_{B,t-1} - L_{B,t}), \\
&\qquad L_{B,t} \in [0, K_{B,t-1}], \quad K^o_{B,t}, B_{B,t} \geq 0, \\
&\qquad \text{the bank-run condition (7).}
\end{aligned}
$$

The bank takes as given the probability distribution of $\{Q_s,\ \hat{\delta}_s,\ \alpha_s,\ \beta c_{i,s}(c_{i,s+1})^{-1} \mid i \in \{i_{S,s}, i_{B,s}\}\}_{s=t}^{\infty}$. The value function, Ω_t, is time-dependent since these exogenous variables for the bank are time-varying. Note that the objective function is the flow-of-fund constraint (6), where V_t is replaced with the first-order condition for bank equities held by the agent $i_{S,t}$.

The first constraint of the bank's maximization problem (10) is the first-order condition for the agent $i_{B,t}$'s bank deposits, in which the definition of the ex-post interest rate in Equation (7) is substituted.[9] The bank internalizes the ex-ante deposit interest rate, $\tilde{R}_t$, through this constraint. The second constraint is the law of motion of capital for the bank. It is assumed that bank does not know the depreciation rate of each infinitesimal unit of capital the bank holds.[10] Thus, the average depreciation rate of capital sold by the bank

[9] In the first constraint, $K_{B,t}(B_{B,t})^{-1}$ is replaced with infinity if $B_{B,t} = 0$.

[10] This assumption will ensure that the bank does not sell a low-quality fraction of used capital selectively in equilibrium. If the bank also had private information, then the adverse selection problem could be worsened by the existence of the bank since the bank does not have an opportunity of investment in new capital and would sell only a low-quality fraction of used capital. Even in this case, the average quality of bank assets

equals $\bar{\delta}$ by the law of large numbers. On the third line are a constraint that the bank cannot sell used capital more than it owns and a non-negativity constraint on bank's choice variables. The last line is the bank-run condition described above.

If there is no equity holder for the bank (i.e., $S_{B,t-1} = 0$), then the bank maximizes the profit from initial public offering of its equities and consumes the profit right away. The profit equals the value of Ω_t. Thus the maximization problem (10) covers this case.[11]

2.3 Definition of an equilibrium

For exposition purpose, this paper first shows the endogenous determination of illiquidity of capital analytically by using the model without the bank and then introduce the bank to the model to show that illiquidity of capital leads to liquidity transformation by the bank. Each exercise needs to define an equilibrium.

Call a set of endogenous variables, $\{c_{i,s},\ x_{i,s},\ k_{i,s},\ k^o_{i,s},\ l_{i,s},\ b_{i,s},\ s_{i,s},\ K_{B,s},\ K^o_{B,s},\ L_{B,s},\ B_{B,s},$ $S_{B,s},\ Q_s,\ \hat{\delta}_s,\ D_s,\ V_s,\ \tilde{R}_s \mid i \in I\}_{s=0}^{\infty}$, contingent on the realization of $\{\alpha_s,\ \Delta_s,\ \phi_{i,s} \mid i \in I\}_{s=0}^{\infty}$ as a contingent plan. Given the set of parameter values, $\{\bar{\delta},\ \phi,\ \beta,\ \zeta,\ \rho_P,\ \rho_U,\ \bar{\alpha},\ \underline{\alpha},\ \bar{\eta}_\alpha,\ \underline{\eta}_\alpha,\ \bar{\Delta},\ \underline{\Delta},$ $\bar{\eta}_\Delta,\ \underline{\eta}_\Delta\}$, and the initial condition on $\{k_{i,-1}, b_{i,-1}, s_{i,-1}, K_{B,-1}, B_{B,-1}, S_{B,-1}, \tilde{R}_{-1} \mid i \in I\}$, an equilibrium for the model with a banking sector is defined as a contingent plan characterized by: the maximization problems (5) and (10) are solved; agents and the bank hold rational expectations; the average depreciation rate of used capital sold in the market is determined by

$$\hat{\delta}_t = \frac{\int_I \int_{\bar{\delta}-\Delta_t}^{\bar{\delta}+\Delta_t} \delta\ l_{i,\delta,t}\ d\delta\ \mu(di) + \bar{\delta}\ L_{B,t}}{\int_I \int_{\bar{\delta}-\Delta_t}^{\bar{\delta}+\Delta_t} l_{i,\delta,t}\ d\delta\ \mu(di) + L_{B,t}}; \tag{11}$$

and the markets for used capital, bank deposits and bank equities clear every period, such

would be public information, given that agents have rational expectations of bank behaviour.

[11]It can be shown that the bank's profit from initial public offering becomes zero in equilibrium.

that

$$\int_I \int_{\bar{\delta}-\Delta_t}^{\bar{\delta}+\Delta_t} k^o_{i,\delta,t} \, d\delta \, \mu(di) + K^o_{B,t} = \int_I \int_{\bar{\delta}-\Delta_t}^{\bar{\delta}+\Delta_t} l_{i,\delta,t} \, d\delta \, \mu(di) + L_{B,t}, \tag{12}$$

$$\int_I b_{i,t} \, \mu(di) = B_{B,t}, \tag{13}$$

$$\int_I s_{i,t} \, \mu(di) = S_{B,t}. \tag{14}$$

An equilibrium for the model without a banking sector is a contingent plan characterized by: the maximization problem (5) is solved with $b_{i,t} = s_{i,t} = 0$ for all i and t; agents hold rational expectations; and Equations (11) and (12) are satisfied with $K^o_{B,t} = L_{B,t} = 0$ for all t.

3 Aggregate equilibrium conditions

3.1 Shock processes

The dynamic analysis of the model with a banking sector will investigate business cycles driven by each type of the two shocks, α_t and Δ_t. Set $\bar{\Delta} = \underline{\Delta}$ when analyzing productivity-driven business cycles, and set $\bar{\alpha} = \underline{\alpha}$, when analyzing information-driven business cycles. With these assumptions, the number of possible states in the next period becomes two every period, which will simplify the bank's problem about whether the bank should take the risk of a bank-run in the next period, or not.

3.2 Agent's behaviour

Call agents with $\phi_{i,t} = \phi$ as 'productive' and those with $\phi_{i,t} = 0$ as 'unproductive'. Suppose that the following conditions hold:

$$\phi > (1 - \hat{\delta}_t)Q_t^{-1}, \tag{15}$$

$$1 > E_t \left[\frac{\beta c_{i,t} R_{t+1}}{c_{i,t+1}} \;\middle|\; \phi_{i,t} = \phi \right], \tag{16}$$

$$(1 + \zeta)V_t > E_t \left[\frac{\beta c_{i,t}(D_{t+1} + V_{t+1})}{c_{i,t+1}} \;\middle|\; \phi_{i,t} = \phi \right], \tag{17}$$

$$1 = E_t \left[\frac{\beta c_{i,t} R_{t+1}}{c_{i,t+1}} \;\middle|\; \phi_{i,t} = 0 \right], \tag{18}$$

$$(1 + \zeta)V_t = E_t \left[\frac{\beta c_{i,t}(D_{t+1} + V_{t+1})}{c_{i,t+1}} \;\middle|\; \phi_{i,t} = 0 \right]. \tag{19}$$

The left-hand side of the first condition is the productivity of investment in new capital, and the right-hand side is the quantity of capital net depreciation that an agent can purchase from the market with a unit of goods. This condition implies that investment in new capital is more profitable than purchasing used capital from the market. The other conditions imply that the rate of returns on the productive's investment into new capital dominates the rates of returns on bank securities and that the unproductive are indifferent between consumption and holding bank securities. With these conditions, it is possible to show that $x_{i,t} > 0$ and $k^o_{i,t} = b_{i,t} = s_{i,t} = 0$ for the productive, that $x_{i,t} = 0$ for the unproductive, and that agents $i_{S,t}$ and $i_{B,t}$ are unproductive. These conditions will be verified in the numerical examples of equilibria considered below.

Each agent sells used capital if selling used capital has a higher rate of returns than keeping the used capital until the next period. Thus:

$$l_{i,\delta,t} = \begin{cases} k_{i,\delta,t-1}, & \text{if } Q_t \geq \lambda_{i,t}(1-\delta), \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

where $\lambda_{i,t}$ is the shadow value of capital net depreciation at the end of period t (i.e., $k_{i,t}$), which is given by the Lagrange multiplier for the law of motion of capital in the maximization problem (5).

It can be shown that, given $x_{i,t} > 0$ for the productive, the productive's shadow value of capital net depreciation equals the marginal cost of producing new capital. Thus:

$$\lambda_{P,t} = \phi^{-1}, \tag{21}$$

where $\lambda_{P,t}$ denotes the shadow value of capital net depreciation, $\lambda_{i,t}$, for the productive. On the other hand, the unproductive's shadow value of capital net depreciation does not necessarily equal the marginal acquisition cost of capital net depreciation from the market, $Q_t(1-\hat{\delta}_t)^{-1}$, since $k^o_{i,t}$ can be zero if the unproductive choose to store their wealth only through bank securities. It can be shown that:

$$\begin{cases} \lambda_{U,t} = Q_t(1-\hat{\delta}_t)^{-1}, & \text{if } k^o_{i,t} > 0 \text{ for the unproductive,} \\ k^o_{i,t} = 0 \text{ for the unproductive,} & \text{if } \lambda_{U,t} < Q_t(1-\hat{\delta}_t)^{-1}, \end{cases} \tag{22}$$

in equilibrium, where $\lambda_{U,t}$ denotes the shadow value of capital net depreciation, $\lambda_{i,t}$, for the unproductive. The second line implies that the unproductive choose $k^o_{i,t} = 0$ if their shadow value of capital net depreciation is lower than the marginal acquisition cost of capital net depreciation.

Given Equation (20), the lower bound for the depreciation rate of used capital sold by each agent is determined by:

$$\tilde{\delta}_{i,t} = \max\left\{\bar{\delta} - \Delta_t,\ \min\left\{\bar{\delta} + \Delta_t,\ 1 - \frac{Q_t}{\lambda_{i,t}}\right\}\right\}. \tag{23}$$

The maximum and the minimum operators in Equation (23) ensure that $\tilde{\delta}_{i,t}$ is within the range of the uniform distribution of δ. Denote the values of $\tilde{\delta}_{i,t}$ for the productive and the unproductive by $\tilde{\delta}_{P,t}$ and $\tilde{\delta}_{U,t}$, respectively. Given Equations (21)-(23), these values are defined as:

$$\tilde{\delta}_{P,t} = \max\left\{\bar{\delta} - \Delta_t,\ \min\left\{\bar{\delta} + \Delta_t,\ 1 - \phi Q_t\right\}\right\}, \tag{24}$$

$$\tilde{\delta}_{U,t} = \max\left\{\bar{\delta} - \Delta_t,\ \min\left\{\bar{\delta} + \Delta_t,\ 1 - \frac{Q_t}{\lambda_{U,t}}\right\}\right\}. \tag{25}$$

Apply the envelop theorem to the maximization problem (5) to find that:

$$\lambda_{i,t} = E_t\left[\frac{\beta c_{i,t}}{c_{i,t+1}}\left(\alpha_{t+1} + \lambda_{i,t+1}\int_{\bar{\delta}-\Delta_{t+1}}^{\tilde{\delta}_{i,t+1}} \frac{1-\delta}{2\Delta_{t+1}}\, d\delta + Q_{t+1}\int_{\tilde{\delta}_{i,t+1}}^{\bar{\delta}+\Delta_{t+1}} \frac{1}{2\Delta_{t+1}}\, d\delta\right)\right]. \tag{26}$$

This dynamic optimization condition implies the following decision rule for each agent:

$$c_{i,t} = (1-\beta)w_{i,t}, \tag{27}$$

$$\lambda_{i,t}k_{i,t} + b_{i,t} + (1+\zeta)V_t s_{i,t} = \beta w_{i,t}, \tag{28}$$

where $w_{i,t}$ is the agent's net-worth defined by

$$w_{i,t} \equiv \left(\alpha_t + \lambda_{i,t}\int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{i,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t\int_{\tilde{\delta}_{i,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta\right) k_{i,t-1} + R_t b_{i,t-1} + (D_t + V_t)s_{i,t-1}. \tag{29}$$

In the definition of net-worth, the fraction of used capital sold by the agent is evaluated by the market price of used capital, Q_t, while the fraction of used capital kept by the agent is evaluated by the shadow value of capital net depreciation, $\lambda_{i,t}$, for the agent. In Equation (28), capital net depreciation at the end of the period, $k_{i,t}$, is also evaluated by the shadow value of capital net depreciation for the agent.

Aggregation of Equation (28) for each type of agent leads to the following aggregate decision rules:

$$\frac{K_{P,t}}{\phi} = \beta \left\{ \left[\alpha_t + \frac{1}{\phi} \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t \int_{\tilde{\delta}_{P,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta \right] [\rho_P K_{P,t-1} + (1-\rho_U) K_{U,t-1}] \right. \\ \left. + (1-\rho_U)[R_t B_{U,t-1} + (D_t + V_t) S_{U,t-1}] \right\}, \quad (30)$$

$$\lambda_{U,t} K_{U,t} + B_{U,t} + (1+\zeta) V_t S_{U,t} = \beta \left\{ \left[\alpha_t + \lambda_{U,t} \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{U,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t \int_{\tilde{\delta}_{U,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta \right] \right. \\ \left. \cdot [(1-\rho_P) K_{P,t-1} + \rho_U K_{U,t-1}] + \rho_U [R_t B_{U,t-1} + (D_t + V_t) S_{U,t-1}] \right\}, \quad (31)$$

where $K_{P,t} = \int_{\{i|\phi_{i,t}=\phi\}} k_{i,t}\ \mu(di)$, $K_{U,t} = \int_{\{i|\phi_{i,t}=0\}} k_{i,t}\ \mu(di)$, $B_{U,t} = \int_{\{i|\phi_{i,t}=0\}} b_{i,t}\ \mu(di)$, and $S_{U,t} = \int_{\{i|\phi_{i,t}=0\}} s_{i,t}\ \mu(di)$.

Also, the law of motion of capital in the maximization problem (5) implies that:

$$\phi X_{P,t} = K_{P,t} - \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{1-\delta}{2\Delta_t}\, d\delta\, [\rho_P K_{P,t-1} + (1-\rho_U) K_{U,t-1}], \quad (32)$$

$$(1-\hat{\delta}_t) K^o_{U,t} = K_{U,t} - \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{U,t}} \frac{1-\delta}{2\Delta_t}\, d\delta\, [(1-\rho_P) K_{P,t-1} + \rho_U K_{U,t-1}], \quad (33)$$

where $X_{P,t} = \int_{\{i|\phi_{i,t}=\phi\}} x_{i,t}\ \mu(di)$ and $K^o_{U,t} = \int_{\{i|\phi_{i,t}=0\}} k^o_{i,t}\ \mu(di)$.

3.3 Bank's behaviour

The representative bank in the competitive banking sector solves the maximization problem (10), given that the number of possible states in the next period is two every period as assumed above. Denote the lower value of $\alpha_{t+1}+Q_{t+1}$ by $\underline{\omega}_{t+1}$ and the higher value by $\bar{\omega}_{t+1}$. The solution to the maximization problem (10) implies the following proposition.

Proposition 3.1. 'Prob($\bar{\omega}_{t+1}$)' denotes the conditional probability that $\alpha_{t+1}+Q_{t+1}=\bar{\omega}_{t+1}$, given the values of period-t variables. In equilibrium with Inequalities (15)-(17), the total value of the bank for equity holders is given by:

$$\Omega_t(K_{B,t-1}, B_{B,t-1}, \tilde{R}_{t-1}) = \begin{cases} \left[\alpha_t + \lambda_{B,t}(1-\bar{\delta})\right] K_{B,t-1} - \tilde{R}_{t-1} B_{B,t-1}, & \text{if } \tilde{R}_{t-1} B_{B,t-1} \leq (\alpha_t + Q_t) K_{B,t-1}, \\ 0, & \text{if } \tilde{R}_{t-1} B_{B,t-1} > (\alpha_t + Q_t) K_{B,t-1}, \end{cases} \quad (34)$$

where

$$\lambda_{B,t} = \max\{\lambda'_{B,t},\ \lambda''_{B,t}\}, \quad (35)$$

$$\lambda'_{B,t} = E_t \left\{ \left. \frac{\beta c_{i,t} \left[\alpha_{t+1} + \frac{Q_{t+1}}{1-\hat{\delta}_{t+1}}(1-\bar{\delta}) - \underline{\omega}_{t+1}\right]}{(1+\zeta) c_{i,t+1}} \right| \phi_{i,t} = 0 \right\} + E_t \left[\left. \frac{\beta c_{i,t}\, \underline{\omega}_{t+1}}{c_{i,t+1}} \right| \phi_{i,t} = 0 \right], \quad (36)$$

$$\lambda''_{B,t} = \text{Prob}(\bar{\omega}_{t+1}) E_t \left\{ \left. \frac{\beta c_{i,t} \left[\alpha_{t+1} + \frac{Q_{t+1}}{1-\hat{\delta}_{t+1}}(1-\bar{\delta}) - \bar{\omega}_{t+1}\right]}{(1+\zeta) c_{i,t+1}} \right| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right\} + E_t \left[\left. \frac{\beta c_{i,t} (\alpha_{t+1} + Q_{t+1})}{c_{i,t+1}} \right| \phi_{i,t} = 0 \right]. \quad (37)$$

Also:

$$\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}, \quad \text{if } \lambda'_{B,t} > \lambda''_{B,t}, \tag{38}$$

$$\tilde{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}, \quad \text{if } \lambda'_{B,t} < \lambda''_{B,t}. \tag{39}$$

Proof: See Appendix A.

Equation (34) implies that the total value of the bank for equity holders is determined by the shadow value of capital net depreciation for the bank, which is denoted by $\lambda_{B,t}$. Equations (35)-(39) indicate that, to maximize the shadow value of capital net depreciation, the bank compares the payoffs from the two levels of bank deposits, $\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ and $\tilde{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$, in equilibrium. The bank focuses on these two options, since it must pay a higher rate of returns on bank equities than bank deposits due to the bank-equity holding cost for agents, ζ, and prefers to finance its assets through bank deposits as much as possible. The bank chooses $\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$, if increasing $B_{B,t}$ above this level would reduce the price of bank equities too much by making a bank run possible in the next period. The value of $\lambda_{B,t}$ equals the larger value between $\lambda'_{B,t}$ and $\lambda''_{B,t}$, which are the shadow values of capital net depreciation when $\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ and when $\tilde{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$, respectively.

It is possible to show that $\lambda_{B,t} = Q_t(1 - \hat{\delta}_t)^{-1}$ if $K^o_{B,t} > 0$ and that $L_{B,t} = 0$ if $\hat{\delta}_t > \bar{\delta}$ and $K^o_{B,t} > 0$ in equilibrium. See Appendix A for the proof of these results. The first result implies that the shadow value of capital net depreciation for the bank must equal the marginal acquisition cost of capital net depreciation if the bank buys used capital. The second result implies that it is not profitable for the bank to sell the bank's own used capital without knowing the true quality of each unit of capital when the bank buys low-quality used capital from the market. If $L_{B,t} = 0$, then the law of motion of capital for the bank becomes:

$$K_{B,t} = (1 - \hat{\delta}_t)K^o_{B,t} + (1 - \bar{\delta})K_{B,t-1}. \tag{40}$$

3.4 Aggregate equilibrium conditions

Hereafter, suppose that $\hat{\delta}_t > \bar{\delta}$, $\lambda'_{B,t} > \lambda''_{B,t}$, and $\lambda_{B,t} = Q_t(1-\hat{\delta}_t)^{-1}$, so that $L_{B,t} = 0$, $K^o_{B,t} > 0$ and $\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ in equilibrium. Thus, the bank conducts liquidity transformation and prevents a bank run by controlling the supply of bank deposits. These conditions will be verified in equilibria considered below.

Given $L_{B,t} = 0$, Equation (11) implies that the average depreciation rate of used capital sold in the market, $\hat{\delta}_t$, is determined by:

$$\hat{\delta}_t = \frac{\int_{\tilde{\delta}_{P,t}}^{\bar{\delta}+\Delta_t} \frac{\delta}{2\Delta_t}\, d\delta\, [\rho_P K_{P,t-1} + (1-\rho_U)K_{U,t-1}] + \int_{\tilde{\delta}_{U,t}}^{\bar{\delta}+\Delta_t} \frac{\delta}{2\Delta_t}\, d\delta\, [(1-\rho_P)K_{P,t-1} + \rho_U K_{U,t-1}]}{\int_{\tilde{\delta}_{P,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta\, [\rho_P K_{P,t-1} + (1-\rho_U)K_{U,t-1}] + \int_{\tilde{\delta}_{U,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta\, [(1-\rho_P)K_{P,t-1} + \rho_U K_{U,t-1}]}. \tag{41}$$

Also, Equations (32)-(33) and (40) (the laws of motion of capital for the productive, the unproductive, and the bank, in order) and Equation (12) (the market clearing condition for used capital) imply that the aggregate law of motion of capital for the economy is:

$$K_{P,t} + K_{U,t} + K_{B,t} = \phi X_{P,t} + (1 - \bar{\delta})(K_{P,t-1} + K_{U,t-1} + K_{B,t-1}). \tag{42}$$

Given Inequalities (15)-(17), $\lambda'_{B,t} > \lambda''_{B,t}$ and $\hat{\delta}_t > \bar{\delta}$, the equilibrium dynamics of $\{K_{P,t}$, $X_{P,t}$, $K_{U,t}$, $K^o_{U,t}$, $B_{U,t}$, $\lambda_{U,t}$, $(D_t + V_t)S_{U,t-1}$, $V_t S_{U,t}$, $K_{B,t}$, $K^o_{B,t}$, Q_t, R_t, $\tilde{R}_t$, $\hat{\delta}_t$, $\tilde{\delta}_{P,t}$, $\tilde{\delta}_{U,t}\}$ is sequentially determined by Equations (18)-(19), (24)-(25), (30)-(33), and (40)-(42), and:

$$(D_t + V_t)S_{U,t-1} = \left[\alpha_t + \frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t}\right] K_{B,t-1} - \tilde{R}_{t-1}B_{U,t-1}, \tag{43}$$

$$\frac{Q_t}{1-\hat{\delta}_t} = E_t \left\{ \frac{\beta c_{i,t}\left[\alpha_{t+1} + \frac{Q_{t+1}}{1-\hat{\delta}_{t+1}}(1-\bar{\delta}) - \underline{\omega}_{t+1}\right]}{(1+\zeta)c_{i,t+1}} + \frac{\beta c_{i,t}\ \underline{\omega}_{t+1}}{c_{i,t+1}} \middle| \phi_{i,t} = 0 \right\}, \tag{44}$$

$$\tilde{R}_t B_{U,t} = \underline{\omega}_{t+1} K_{B,t}, \tag{45}$$

$$R_t = \tilde{R}_{t-1}. \tag{46}$$

Equations (43), (44), (45), and (46) are derived from: Equation (34) and $\lambda_{B,t} = Q_t(1 - \hat{\delta}_t)^{-1}$; Equations (35)-(36) and $\lambda_{B,t} = Q_t(1-\hat{\delta}_t)^{-1}$; Equation (38); and Equation (7), in order. The market clearing conditions for bank equities and bank deposits, $S_{B,t} = S_{U,t}$ and $B_{B,t} = B_{U,t}$, respectively, are substituted in Equations (43) and (45).

4 Endogenous illiquidity of assets

This section shows the closed form for the equilibrium dynamics of the model without a banking sector to show endogenous determination of illiquidity of used capital analytically. This is for exposition purpose, since the closed form for the dynamics of the model with a banking sector cannot be obtained. The intuition behind the results of the model without a banking sector is shared with the model with a banking sector. In this section, $x_{i,t} = 0$ and $k^o_{i,t} > 0$ for the unproductive, since there is no supply of bank securities and the unproductive can only store their wealth through buying used capital. Given Inequality (15), $x_{i,t} > 0$ and $k^o_{i,t} = 0$ for the productive.

Aggregate equilibrium conditions are identical with the model with a banking sector except that $K_{B,t} = B_{U,t} = S_{U,t} = 0$ for all t. Given $k^o_{i,t} > 0$ for the unproductive, Equation

(22) implies that $\lambda_{U,t} = Q_t(1 - \hat{\delta}_t)^{-1}$. Then, Equations (24)-(25) lead to:

$$\tilde{\delta}_{P,t} = \max\left\{\bar{\delta} - \Delta_t,\ 1 - \phi Q_t\right\} < \tilde{\delta}_{U,t} = \hat{\delta}_t, \tag{47}$$

given Inequality (15).[12]

Given the values of α_t, Δ_t, $K_{P,t-1}$ and $K_{U,t-1}$ and that $K_{B,t} = B_{U,t} = S_{U,t} = 0$ for all t, Equations (31), (32) and (42) imply that the equilibrium values of $\hat{\delta}_t$ and Q_t in each period are determined by Equation (41) and:

$$\begin{aligned} g(\hat{\delta}_t, Q_t, \alpha_t, \Delta_t, \theta_{K,t-1}) \equiv \frac{Q_t}{1-\hat{\delta}_t}\left[(1-\bar{\delta})(1+\theta_{K,t-1}) - \theta_{K,t-1}\int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{1-\delta}{2\Delta_t}\, d\delta\right] \\ - \beta\left(\alpha_t + \frac{Q_t}{1-\hat{\delta}_t}\int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{U,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t \int_{\tilde{\delta}_{U,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta\right) = 0, \end{aligned} \tag{48}$$

where

$$\theta_{K,t-1} \equiv \frac{\rho_P K_{P,t-1} + (1-\rho_U)K_{U,t-1}}{(1-\rho_P)K_{P,t-1} + \rho_U K_{U,t-1}}, \tag{49}$$

and the values of $\tilde{\delta}_{P,t}$ and $\tilde{\delta}_{U,t}$ are as shown in Equation (47). Then, the values of $K_{P,t}$ and $K_{U,t}$ are determined by Equations (30)-(31), given $\lambda_{U,t} = Q_t(1-\hat{\delta}_t)^{-1}$ and $B_{U,t} = S_{U,t} = 0$ for all t. The first term of the function g is the shadow value of aggregate capital net depreciation that the unproductive must hold at the end of the period in equilibrium and the second term is the fraction of the unproductive's aggregate net-worth that is spent on used capital. Both terms are normalized by the aggregate net-worth of the unproductive.

[12]Note that $\tilde{\delta}_{P,t} < \hat{\delta}_t \leq \bar{\delta} + \Delta_t$.

4.1 *Illiquidity of capital due to adverse selection*

The result that $\tilde{\delta}_{U,t} = \hat{\delta}_t$ indicates that the quality of used capital sold by the unproductive is always worse than the average quality of used capital sold in the market. This is adverse selection. In equilibrium, the adverse selection raises the average depreciation rate of used capital sold in the market, $\hat{\delta}_t$, which leads to $\hat{\delta}_t > \bar{\delta}$. This result can be confirmed by substituting $\tilde{\delta}_{U,t} = \hat{\delta}_t$ in Equation (41).

The result that $\hat{\delta}_t > \bar{\delta}$ implies that each agent's used capital as a whole is undervalued in the market, which can be shown by substituting $\lambda_{U,t} = Q_t(1-\hat{\delta}_t)^{-1}$ into Equation (26):

$$Q_t = (1-\hat{\delta}_t)E_t\left[\frac{\beta c_{i,t}}{c_{i,t+1}}\left(\alpha_{t+1} + \lambda_{i,t+1}\int_{\bar{\delta}-\Delta_{t+1}}^{\tilde{\delta}_{i,t+1}} \frac{1-\delta}{2\Delta_{t+1}}\,d\delta + Q_{t+1}\int_{\tilde{\delta}_{i,t+1}}^{\bar{\delta}+\Delta_{t+1}} \frac{1}{2\Delta_{t+1}}\,d\delta\right)\middle|\,\phi_{i,t}=0\right]. \tag{50}$$

This equation shows that the market price of used capital, Q_t, depends on $\hat{\delta}_t$. The true average value of each agent's used capital is obtained by replacing $\hat{\delta}_t$ with $\bar{\delta}$ on the right-hand side of the equation, given $\lambda_{i,t+1}$ and Q_{t+1} in the next period. Thus, the result that $\hat{\delta}_t > \bar{\delta}$ implies that the market value of used capital is lower than the true average value of used capital held by each agent.

In this paper, define illiquidity of an asset as undervaluation of the asset in the market. The degree of illiquidity of each agent's used capital as a whole is measured by the difference between $\hat{\delta}_t$ and $\bar{\delta}$. Hereafter, take $\hat{\delta}_t$ as the indicator of illiquidity of used capital, since $\bar{\delta}$ is fixed.

4.2 *Response of illiquidity of capital to productivity shocks*

Sections 4.2 and 4.3 will show that both a negative productivity shock and an increase in the degree of asymmetric information can cause an increase in illiquidity of assets (used capital) represented by $\hat{\delta}_t$ and a decline in the market price of assets, Q_t, as occurred during the financial crisis since 2007.

Equations (41) and (48) have the following characteristics:

Lemma 4.1. Equation (48) is downward-sloping curves on the $(Q_t, \hat{\delta}_t)$ plane, given the values of α_t, Δ_t and $\theta_{K,t-1}$. Equation (41) is also downward-sloping, if $\tilde{\delta}_{P,t} = 1 - \phi Q_t$, and is a flat line, if $\tilde{\delta}_{P,t} = \bar{\delta} - \Delta_t$. Equation (48) has a steeper slope than Equation (41) at the intersection of the two curves, if β is sufficiently close to 1.

Proof: See Appendix B.

Figure 1 draws Equations (41) and (48) on the $(Q_t, \hat{\delta}_t)$ plane and shows how a decline in α_t makes them shift.[13] It is obvious that Equation (41) does not shift. Equation (48) shifts inward, since a decline in α_t reduces the aggregate income of the unproductive, which lowers Q_t through a decreased aggregate spending on used capital. If $\tilde{\delta}_{P,t} = 1 - \phi Q_t$ in the new equilibrium, then the equilibrium shifts along a downward-sloping part of Equations (41), as shown in the figure. In this case, a decline in the market price of used capital, Q_t, due to a negative productivity shock discourages the productive from selling high-quality used capital in the market (i.e., a rise in $\tilde{\delta}_{P,t}$), which increases the average depreciation rate of used capital sold in the market, $\hat{\delta}_t$.

4.3 Response of illiquidity of capital to shocks to the degree of asymmetric information

Figure 2 shows the effect of a rise in Δ_t, which increases the degree of asymmetric information. An increase in Δ_t makes Equation (41) shift upward unambiguously, since an expanded range of depreciation rates of used capital lets each agent sell the increased low-quality fraction of used capital while keeping the increased high-quality fraction of used capital, which raises $\hat{\delta}_t$ through worsened adverse selection.

[13] Due to the log utility function, Equations (41) and (48) are valid irrespective of the stochastic process of the shocks in the model.

On the other hand, the following lemma implies that the direction of the shift in Equation (48) is ambiguous:

Lemma 4.2. $\frac{\partial g}{\partial \Delta_t} > 0$, if $\tilde{\delta}_{P,t}$ is sufficiently close to $\hat{\delta}_t$. $\frac{\partial g}{\partial \Delta_t} < 0$, if $\tilde{\delta}_{P,t} \leq \bar{\delta}$.[14]

Proof: See Appendix C.

Since $\frac{\partial g}{\partial Q_t} > 0$, Equation (48) shifts inward in the first case ($\frac{\partial g}{\partial \Delta_t} > 0$) and outward in the second case ($\frac{\partial g}{\partial \Delta_t} < 0$), in response to an increase in Δ_t.[15] The top panel of Figure 2 shows the first case and the bottom panel shows the second case. In the first case, $\hat{\delta}_t$ increases and Q_t decreases. Since Equation (41) implies that $\hat{\delta}_t$ is close to $\bar{\delta} + \Delta_t$ if $\tilde{\delta}_{P,t}$ is close to $\hat{\delta}_t$, this result indicates that an increase in the degree of asymmetric information causes a simultaneous deterioration of illiquidity of used capital and the market price of used capital if adverse selection in the asset market is so severe that the volume of trade in the market is small.

5 Liquidity transformation and bank capital requirements

5.1 Liquidity transformation by the bank

This section analyzes the features of the model with a banking sector. The aggregate decision rules specified by Equations (30)-(31) are useful to explain why agents hold bank securities.

[14] On the balanced growth path, $\tilde{\delta}_{P,t}$ is close to $\hat{\delta}_t$ if ρ_P and $1-\rho_U$ are high, and $\tilde{\delta}_{P,t} \leq \bar{\delta}$ if ρ_P and $1-\rho_U$ are low. See Appendix D for more details.

[15] The intuition for this result is that, when $\frac{\partial g}{\partial \Delta_t} > 0$, a rise in Δ_t reduces the fraction of used capital kept by the productive, $\int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} (1-\delta)/(2\Delta_t)\, d\delta$, in the first term of the function g in Equation (48). As a consequence, the unproductive must absorb a larger amount of used capital, which reduces the price of used capital, Q_t, given the value of $\hat{\delta}_t$. When $\frac{\partial g}{\partial \Delta_t} < 0$, a rise in Δ_t increases the fraction of used capital kept by the productive. Less supply of used capital leads to an increase in Q_t, given the value of $\hat{\delta}_t$. Also, in both cases, a rise in Δ_t increases the second term of the function g in Equation (48) (i.e., the aggregate net-worth of the unproductive), since the unproductive benefit from more opportunities for adverse selection. This effect would increase Q_t. When $\frac{\partial g}{\partial \Delta_t} > 0$, this effect is dominated by the effect of an increased supply of used capital by the productive.

By substituting Equations (43) and (46), Equations (30)-(31) can be rewritten as:

$$\frac{K_{P,t}}{\phi} = \beta \left\{ \left[\alpha_t + \frac{1}{\phi} \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t \int_{\tilde{\delta}_{P,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta \right] \left[\rho_P K_{P,t-1} + (1-\rho_U) K_{U,t-1}\right] \right.$$
$$\left. +(1-\rho_U) \left[\alpha_t + \frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} \right] K_{B,t-1} \right\}, \quad (51)$$

$$\lambda_{U,t} K_{U,t} + B_{U,t} + (1+\zeta) V_t S_{U,t} = \beta \left\{ \left[\alpha_t + \lambda_{U,t} \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{U,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t \int_{\tilde{\delta}_{U,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta \right] \right.$$
$$\left. \cdot \left[(1-\rho_P) K_{P,t-1} + \rho_U K_{U,t-1}\right] + \rho_U \left[\alpha_t + \frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} \right] K_{B,t-1} \right\}. \quad (52)$$

Note that $K_{U,t-1}$ is replaced with $K_{B,t-1}$ as the unproductive shift their portfolio from used capital to bank securities. Thus, comparing the coefficients to $K_{U,t-1}$ and $K_{B,t-1}$ clarifies the benefit of holding bank securities. The following proposition holds.

Proposition 5.1. If Inequality (15) holds and $\hat{\delta}_t > \bar{\delta}$, then:

$$\frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} > \frac{1}{\phi} \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t \int_{\tilde{\delta}_{P,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta. \quad (53)$$

If $\hat{\delta}_t > \bar{\delta}$ and $\lambda_{U,t} = Q_t(1-\hat{\delta}_t)^{-1}$, then:

$$\frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} < \lambda_{U,t} \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{U,t}} \frac{1-\delta}{2\Delta_t}\, d\delta + Q_t \int_{\tilde{\delta}_{U,t}}^{\bar{\delta}+\Delta_t} \frac{1}{2\Delta_t}\, d\delta. \quad (54)$$

Proof: See Appendix E.

Inequality (53) implies that the value of the productive's net-worth increases as $K_{U,t-1}$ is replaced with $K_{B,t-1}$. Thus, agents can increase investments in new capital by storing wealth through bank securities when they are unproductive and selling them when they are productive. Note that, when agents sell bank securities, they transfer a share of the whole used capital the bank holds. Since idiosyncratic depreciation rates of the bank's whole used capital cancel each other out, the value of the bank's used capital that backs bank securities becomes public information, which makes bank securities free from adverse selection. Hence, as indicated by Equations (43), the value of bank securities reflects the shadow value of bank's whole used capital, i.e., $[\alpha_t + Q_t(1-\hat{\delta}_t)^{-1}(1-\bar{\delta})]K_{B,t-1}$, instead of the liquidation value of the used capital, $(\alpha_t+Q_t)K_{B,t-1}$. Sellers of bank securities can obtain a fair amount of goods for bank securities.

On the other hand, Inequality (54) indicates that there is a case where holding bank securities is ex-post costly for the unproductive if they remain unproductive in the next period, since they lose the opportunity to sell a low-quality fraction of used capital at an overvalued market price. Overall, the unproductive hold bank securities if the expected benefit of holding liquid bank securities for increasing investment into new capital dominates the expected cost of losing the opportunity to sell low-quality used capital at an overvalued market price.[16]

5.2 Comparative statics analysis of introduction of the bank to the economy

Figure 3 compares balanced growth paths with and without the bank. The figure is a numerical example of comparative statics around a set of benchmark parameter values that approximately replicates the post-war sample average of US data on the balanced growth path with the bank.[17] In the figure, the time index, t, is omitted from the notation of each

[16] If the probability for the unproductive to be productive in the next period, $1-\rho_U$, is sufficiently low, then the unproductive do not hold bank securities and financial intermediation does not arise in equilibrium.

[17] The benchmark parameter values are $(\bar{\delta}, \phi, \beta, \zeta, \rho_P, \rho_U) = (0.1, 4.75, 0.99, 0.02, 0.45, 0.55)$, $\bar{\alpha} = \underline{\alpha} = 0.03$, and $\bar{\Delta} = \underline{\Delta} = 0.09$. Suppose the length of a period in the model is a year. For 1948-2007 in U.S., the average real GDP growth rate is 3.4%, the average real interest rate on 3-month treasury bills is 3.9%, and the average ratio of the bank credit of commercial banks to the fixed assets in the economy is 15.0%. These numbers are approximately replicated by the growth rate of aggregate output $(G-1)$, $\tilde{R}-1$, and $K_B/(K_P+K_U+K_B)$, in order. The capital-asset ratio of the bank is around 8% on the balanced growth path in the model, which is the minimum requirement by the Basel agreement. The 10% annual depreciation rate of capital implied by $\bar{\delta}$ is a standard assumption. Rouwenhoust (1995) reports that the equity premium on S&P 500 was 1.99% on average for 1948-1992. The equity premium on bank equities in the model takes a similar value. The

variable. The deposit interest rate, $\tilde{R}$, for the model without the bank is a hypothetical rate with no supply of bank deposits. The figure shows balanced growth paths under various values of ζ.

Figure 3 illustrates that, given parameter values, introduction of the bank to the economy increases $\tilde{R}$. This result is consistent with the analytical result shown in Section 5.1, that agents can increase investments in new capital by storing their wealth through bank securities when they are unproductive and selling them when they are productive. Since bank securities let agents suffer less from illiquidity of their assets when they are productive, the expected consumption in the case of becoming productive increases, which leads to a decline in the stochastic discount factor, $\beta c_{i,t}(c_{i,t+1})^{-1}$, for the unproductive and thus a rise in $\tilde{R}$.

Despite this positive effect of bank securities on the productive who used to be unproductive, Figure 3 shows that the gross rate of growth of aggregate output, which is denoted by

$$G_t \equiv \frac{Y_t}{Y_{t-1}}, \quad \text{where } Y_t \equiv \int_I y_{i,t}\, \mu(di), \tag{55}$$

does not necessarily increase with introduction of the bank to the economy in the long run. Note that introduction of the bank to the economy leads to a decline in the market price of used capital, Q, through a drop in the stochastic discount factor for the unproductive. A decline in Q discourages agents from selling high-quality capital, which leads to a rise in $\tilde{\delta}_P$ and $\tilde{\delta}_U$ through Equation (23). This effect raises $\hat{\delta}$. A resulting increase in illiquidity of used capital reduces investments in new capital by the productive who continue to be productive from the previous period, since these agents do not hold bank securities and have to suffer from worsened undervaluation of used capital they hold. The figure shows that this negative

data sources for the first three sample averages are NIPA data from the BEA and financial data from the Federal Reserve Board. Note that $\rho_P = 1 - \rho_U$, which implies that the arrival of the opportunity to produce new capital is i.i.d. for each agent. This assumption is set to reduce the dimension of the parameter space.

effect on the productive who continue to be productive from the previous period dominates the positive effect of bank securities on the productive who used to be unproductive, if the marginal bank-equity holding cost, ζ, is sufficiently high. This is because the bank-equity holding cost is incurred only by the unproductive and this cost reduces the net positive effect of bank securities on the productive who used to be unproductive.

5.3 Bank capital requirements: dynamic analysis

Equations (19) and (43)-(45) imply that the capital-asset ratio of the bank is given by:

$$\begin{aligned}\frac{V_t S_{B,t}}{B_{B,t}+V_t S_{B,t}} &= \frac{E_t\left\{\frac{\beta c_{i,t}}{(1+\zeta)c_{i,t+1}}\left[\alpha_{t+1}+\frac{Q_{t+1}(1-\bar{\delta})}{1-\hat{\delta}_{t+1}}-\underline{\omega}_{t+1}\right]\Big|\,\phi_{i,t}=0\right\}K_{B,t}}{Q_t(1-\hat{\delta}_t)^{-1}K_{B,t}} \\ &= \frac{1-\hat{\delta}_t}{Q_t}E_t\left\{\frac{\beta c_{i,t}}{(1+\zeta)c_{i,t+1}}\left[\frac{Q_{t+1}(\hat{\delta}_{t+1}-\bar{\delta})}{1-\hat{\delta}_{t+1}}+(\alpha_{t+1}+Q_{t+1}-\underline{\omega}_{t+1})\right]\Bigg|\,\phi_{i,t}=0\right\}. \quad (56)\end{aligned}$$

Note that the denominator of the ratio, the value of bank assets, equals the total value of liabilities, $B_{B,t}+V_t S_{B,t}$, in the balance sheet of the bank.

The first line of Equation (56) shows that the capital-asset ratio of the bank depends on the present discounted value of the difference between the shadow value of the bank's used capital, $[\alpha_{t+1}+Q_{t+1}(1-\hat{\delta}_{t+1})^{-1}(1-\bar{\delta})]K_{B,t}$, and the borrowing limit on bank deposits, i.e., the worst possible liquidation value of the bank's used capital in the next period, $\underline{\omega}_{t+1}K_{B,t}$. The present discounted value of this difference must be financed through public equities.

The second line of Equation (56) implies that the difference is positive and can be decomposed into two factors. First, illiquidity of capital causes a gap between the shadow value and the realized liquidation value of the bank's used capital, which appears as $Q_{t+1}(\hat{\delta}_{t+1}-\bar{\delta})(1-\hat{\delta}_{t+1})^{-1}$ in Equation (56). Note that this is positive by $\hat{\delta}_{t+1}>\bar{\delta}$. Second, since the liquidation value of the bank's used capital fluctuates, the realized liquidation value of the bank's used capital can be more than the worst possible liquidation value. The difference between the two appears as $\alpha_{t+1} + Q_{t+1} - \underline{\omega}_{t+1}$ in Equation (56). The present discounted value of the difference is positive by the definition of $\underline{\omega}_{t+1}$.

Call the first factor as 'illiquidity factor' and the second factor as 'market-price factor'. To illustrate the effects of these factors on equilibrium dynamics of the capital-asset ratio of the bank, Figures 4 and 5 show sample paths of the dynamics of the model driven by changes in α_t and Δ_t, respectively. See Appendix F for the numerical solution method. The stochastic process of α_t is set so that the growth rate of output is around 4% in booms and around 2% in recessions, on average. The stochastic process of Δ_t is set so that Δ_t fluctuates symmetrically around the benchmark value specified in Section 5.2 and its upper value ($\bar{\Delta}$) takes the maximum value that makes the lower bound of the range of depreciation rates equal to zero. For both processes, the transition probabilities of the shocks are set so that the expected durations of booms and recessions are 4 years, given that the length of a period in the model is interpreted as a year.[18] The parameters except for the shock parameter in each figure take the benchmark values specified in Section 5.2. Each figure shows the sample path when the shock parameter keeps changing its value every 4 periods for a sufficiently long time.

Figure 4 indicates that the capital-asset ratio of the bank is pro-cyclical when business cycles are driven by productivity shocks. This result is due to the market-price factor. When a positive productivity shock hits the economy, the expected income from used capital increases since the shock is persistent. This effect raises the expected value of Q_{t+1} through Equation (44) while positive productivity shocks hit the economy. This effect in turn increases the gap between the expected realized liquidation value of bank's used capital and its worst possible value in the next period. Thus, the capital-asset ratio of the bank rises during booms. It falls during recessions by the same mechanism that works in the opposite

[18]In Figure 4, $\bar{\alpha} = 0.0306$, $\underline{\alpha} = 0.0294$ and $\bar{\Delta} = \underline{\Delta} = 0.09$. In Figure 5, $\bar{\Delta} = 0.1$, $\underline{\Delta} = 0.08$ and $\bar{\alpha} = \underline{\alpha} = 0.03$. For both figures, $\eta_{\bar{x}} = \eta_{\underline{x}} = 0.75$ for $x = \alpha, \Delta$.

direction. Hence, the capital-asset ratio of the bank becomes pro-cyclical.

Note that the indicator of illiquidity of used capital, $\hat{\delta}_t$, is counter-cyclical in Figure 4. This is because the pro-cyclical fluctuations in Q_t induce the productive to sell high-quality capital (i.e., a decline in $\tilde{\delta}_{P,t}$) during booms and to keep it (i.e., a rise in $\tilde{\delta}_{P,t}$) during recessions. Even though this illiquidity factor drives the capital-asset ratio of the bank to be counter-cyclical, the market-price factor dominates the illiquidity factor in the numerical example shown in the figure.

In contrast, Figure 5 indicates that the capital-asset ratio of the bank is counter-cyclical when business cycles are driven by changes in the degree of asymmetric information (changes in Δ_t). In this case, a decline in Δ_t reduces adverse selection in the market for used capital, which lowers $\hat{\delta}_t$ and thus illiquidity of used capital. As a consequence, the capital-asset ratio of the bank drops. At the same time, a decline in illiquidity of used capital facilitates the transfer of goods from the unproductive to the productive in exchange for used capital, raising the growth rate of output. By a similar mechanism, an increase in Δ_t causes an increase in the capital-asset ratio of the bank and a decline in the growth rate of output. Hence, the capital-asset ratio of the bank becomes counter-cyclical. While the counter-cyclical movements of $\hat{\delta}_t$ cause pro-cyclical movements of Q_t through Equation (44), which drives the capital-asset ratio of banks to be pro-cyclical, the illiquidity factor dominates the market-price factor in the numerical example shown in the figure.

6 Conclusion

This paper presents a dynamic competitive equilibrium model in which illiquidity of assets arises endogenously due to asymmetric information about asset quality. It is shown that both a negative productivity shock and an increase in the degree of asymmetric information can cause a simultaneous deterioration of illiquidity of assets and the market price of assets, as occurred during the financial crisis since 2007. The model also shows that illiquidity of assets leads to liquidity transformation by banks and that banks must maintain positive bank capital to prevent a self-fulfilling bank run due to illiquidity of bank assets.

The dynamic analysis of the model indicates that, to prevent a bank run, the capital-asset ratio of banks should be linked to illiquidity of bank assets and the volatility of the market price of bank assets. The numerical examples suggest that the equilibrium capital-asset ratio of banks is pro-cyclical during regular business cycles driven by productivity shocks and that it is counter-cyclical when business cycles are driven by changes in the degree of asymmetric information.

While the equilibrium capital-asset ratio of banks in the model is market discipline imposed by rational agents, it can be seen as a benchmark for dynamic bank-capital regulation, since one of the purposes of the regulation is to achieve financial stability by preventing bank runs. Formal analysis of optimal dynamic bank-capital regulation, including the optimal balance between market discipline and regulation, is left for future research.

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Appendices

A Proof of Proposition 3.1

Suppose that Equations (34)-(37) are true. Note that Equation (34) satisfies the bank-run condition (7).

To verify Equation (34), split the constraint set of the maximization problem (10) into three regions: $\tilde{R}_t B_{B,t} \leq \underline{\omega}_{t+1} K_{B,t}$; $\tilde{R}_t B_{B,t} \in (\underline{\omega}_{t+1} K_{B,t}, \bar{\omega}_{t+1} K_{B,t}]$; and $\tilde{R}_t B_{B,t} > \bar{\omega}_{t+1} K_{B,t}$. First of all, any point in the third region, $\tilde{R}_t B_{B,t} > \bar{\omega}_{t+1} K_{B,t}$, is weakly dominated by the supremum in the second region, since the constraint sets become identical between the two cases, but the value of Ω_{t+1} is always 0 in the third region while it can be positive in the second region. Thus, the third region can be ignored.

Use the Lagrange method to solve the maximization problem in the first and the second regions and compare the maximum values of the value function between the two regions, given Ω_{t+1} defined by Equation (34). To make the second region closed, consider the closure of the region as the constraint set and suppose that Ω_{t+1} takes the limit value when $\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$. This makes the function Ω_{t+1} differentiable in each region. This is just for formality, since it will be shown that $\tilde{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$ at optimum in the second region.

It can be shown that $\tilde{R}_t B_{B,t}$ equals the upper bound at optimum in each region:

Lemma A.1. Suppose that Ω_{t+1} is defined by Equation (34). Then, in equilibrium, $\tilde{R}_t B_{B,t}$ equals $\underline{\omega}_{t+1} K_{B,t}$ at optimum in the first region of the maximization problem (10) and $\bar{\omega}_{t+1} K_{B,t}$ at optimum in the second region.

Proof: In the first region, $\tilde{R}_t$ is determined solely by the first constraint of the maximization problem (10) and can be taken as exogenous for the bank. The first constraint implies that $\tilde{R}_t$ must be positive, since agents never choose zero consumption with the time-separable log utility function in equilibrium. The first-order condition with respect to $B_{B,t}$

is:

$$1 - \frac{1}{1+\zeta} E_t \left[\frac{\beta c_{i,t} \tilde{R}_t}{c_{i,t+1}} \middle| \phi_{i,t} = 0 \right] - \bar{\theta}_{rgn1,t} \tilde{R}_t = 0, \tag{57}$$

where $\bar{\theta}_{rgn1,t}$ is the Lagrange multiplier for the upper bound of the first region ($\tilde{R}_t B_{B,t} \leq \underline{\omega}_{t+1} K_{B,t}$). Thus, $\bar{\theta}_{rgn1,t} = \zeta(1+\zeta)^{-1}(\tilde{R}_t)^{-1} > 0$, given $\zeta > 0$ and $\tilde{R}_t > 0$. Hence, $\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ at optimum in the first region.

For the second region, if $K_{B,t} = 0$, then the claim is automatically satisfied since the first constraint of the maximization problem (10) implies that $B_{B,t}$ must be 0. Hereafter suppose $K_{B,t} > 0$ in the second region. In equilibrium, Q_t is alway positive and thus $\underline{\omega}_t > 0$ for all t, since otherwise each agent would demand an infinite amount of used capital, which would violate the market clearing condition for used capital. In the second region, $K_{B,t} > 0$ and $\underline{\omega}_{t+1} > 0$ implies that $B_{B,t} > 0$ and $\tilde{R}_t > 0$, since $B_{B,t}$ must be non-negative by assumption. The first-order conditions with respect to $B_{B,t}$ and $\tilde{R}_t$ in the second region are, respectively:

$$\begin{aligned} 1 - \frac{\text{Prob}(\bar{\omega}_{t+1})}{1+\zeta} E_t \left[\frac{\beta c_{i,t} \tilde{R}_t}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right] + (\underline{\theta}_{rgn2,t} - \bar{\theta}_{rgn2,t}) \tilde{R}_t \\ - \theta_{PC,t} \text{Prob}(\underline{\omega}_{t+1}) E_t \left[\frac{\beta c_{i,t} \underline{\omega}_{t+1} K_{B,t}}{c_{i,t+1} B_{B,t}^2} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \underline{\omega}_{t+1} \right] = 0, \end{aligned} \tag{58}$$

$$\begin{aligned} - \frac{\text{Prob}(\bar{\omega}_{t+1})}{1+\zeta} E_t \left[\frac{\beta c_{i,t} B_{B,t}}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right] + (\underline{\theta}_{rgn2,t} - \bar{\theta}_{rgn2,t}) B_{B,t} \\ + \theta_{PC,t} \text{Prob}(\bar{\omega}_{t+1}) E_t \left[\frac{\beta c_{i,t}}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right] = 0, \end{aligned} \tag{59}$$

where $\bar{\theta}_{rgn2,t}$ is the Lagrange multiplier for the upper bound of the second region ($\tilde{R}_t B_{B,t} \leq \bar{\omega}_{t+1} K_{B,t}$), $\underline{\theta}_{rgn2,t}$ is the Lagrange multiplier for the lower bound of the second region ($\tilde{R}_t B_{B,t} \geq \underline{\omega}_{t+1} K_{B,t}$), and $\bar{\theta}_{PC,t}$ is the Lagrange multiplier for the first constraint of the maximization problem (10), which is the participation constraint for depositors. The two conditions imply that $\theta_{PC,t} = B_{B,t}$. Substituting this into Equation (58) leads to:

$$(\bar{\theta}_{rgn2,t} - \underline{\theta}_{rgn2,t})\tilde{R}_t = \frac{\zeta \cdot \text{Prob}(\bar{\omega}_{t+1})}{1+\zeta} E_t \left[\frac{\beta c_{i,t} \tilde{R}_t}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right], \tag{60}$$

which in turn indicates that $\bar{\theta}_{rgn2,t} > 0$ and $\underline{\theta}_{rgn2,t} = 0$, given $\zeta > 0$ and $\tilde{R}_t > 0$. Thus, $\tilde{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$ at optimum in the second region. $\square$

Denote the maximum values of the objective function of the maximization problem (10) in the first region and the second regions by Ω'_t and Ω''_t, respectively. By Lemma A.1, $\tilde{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ and $\tilde{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$ at optimum in the first and the second regions, respectively. Also, the first constraint of the maximization problem (10) implies:

$$B_{B,t} = E_t \left[\frac{\beta c_{i,t} \min \left\{ \tilde{R}_t B_{B,t},\ (\alpha_{t+1} + Q_{t+1}) K_{B,t} \right\}}{c_{i,t+1}} \middle| \phi_{i,t} = 0 \right]. \tag{61}$$

Substituting these equations into the objective function of the maximization problem (10) with Ω_{t+1} defined by Equation (34) leads to:

$$\Omega'_t = \alpha_t K_{B,t-1} + Q_t (L_{B,t} - K^o_{B,t}) + \lambda'_{B,t} K_{B,t} - R_t B_{B,t-1}, \tag{62}$$

$$\Omega''_t = \alpha_t K_{B,t-1} + Q_t (L_{B,t} - K^o_{B,t}) + \lambda''_{B,t} K_{B,t} - R_t B_{B,t-1}. \tag{63}$$

Note that the values of Ω'_t and Ω''_t are maximized with the constraints on $K^o_{B,t}$ and $L_{B,t}$ in the maximization problem (10):

$$K_{B,t} = (1 - \hat{\delta}_t) K^o_{B,t} + (1 - \bar{\delta})(K_{B,t-1} - L_{B,t}), \tag{64}$$

$$L_{B,t} \in [0, K_{B,t-1}), \quad K^o_{B,t} \geq 0. \tag{65}$$

Since these constraints are identical between the first and the second regions, $\Omega_t = \Omega'_t$ if $\lambda'_{B,t} \geq \lambda''_{B,t}$ and $\Omega_t = \Omega''_t$ if $\lambda'_{B,t} \leq \lambda''_{B,t}$. Given this result, the maximization problem (10) can be rewritten as:

$$\begin{aligned} \Omega_t = &\max_{\{K^o_{B,t}, L_{B,t}\}} \alpha_t K_{B,t-1} + Q_t(L_{B,t} - K^o_{B,t}) + \lambda_{B,t} K_{B,t} - R_t B_{B,t-1}, \\ &\text{s.t. } K_{B,t} = (1-\hat{\delta}_t)K^o_{B,t} + (1-\bar{\delta})(K_{B,t-1} - L_{B,t}), \\ &\qquad L_{B,t} \in [0, K_{B,t-1}), \quad K^o_{B,t} \geq 0, \\ &\qquad \text{the bank-run condition (7),} \end{aligned} \tag{66}$$

where $\lambda_{B,t} = \max\{\lambda'_{B,t},\ \lambda''_{B,t}\}$. Note that the first constraint of the maximization problem (10) is already incorporated in Equations (62)-(63).

The maximization problem (66) implies that the equilibrium value of $\lambda_{B,t}$ satisfies:

$$\lambda_{B,t} \begin{cases} = Q_t(1-\hat{\delta}_t)^{-1}, & \text{if } K^o_{B,t} > 0, \\ = Q_t(1-\bar{\delta})^{-1}, & \text{if } L_{B,t} \in (0, K_{B,t-1}) \\ \leq Q_t(1-\bar{\delta})^{-1}, & \text{if } L_{B,t} = K_{B,t-1} \\ \in [Q_t(1-\bar{\delta})^{-1},\ Q_t(1-\hat{\delta}_t)^{-1}], & \text{if } K^o_{B,t} = 0 \text{ and } L_{B,t} = 0. \end{cases} \tag{67}$$

Note that this equation indicates that $L_{B,t} = 0$ if $K^o_{B,t} > 0$ and that $K^o_{B,t} = 0$ if $L_{B,t} > 0$, when $\hat{\delta}_t > \bar{\delta}$. Also, if $L_{B,t} = K_{B,t-1}$, then $K_{B,t} = 0$ and $\Omega_t = (\alpha_t + Q_t)K_{B,t-1} - R_t B_{B,t-1}$. Without loss of generality, set $\lambda_{B,t} = Q_t(1-\bar{\delta})^{-1}$ for this case. Note that if $\hat{\delta}_t = \bar{\delta}$, then $\lambda_{B,t} = Q_t(1-\bar{\delta})^{-1}$ in any case. With this definition of $\lambda_{B,t}$, substituting the bank-run condition (7), Equation (64), and Equation (67) for each case of the values of $K^o_{B,t}$ and $L_{B,t}$ into the objective function of the maximization problem (66) verifies the proposition.

B Proof of Lemma 4.1

Denote implicit functions for $\hat{\delta}_t$ implied by Equations (41) and (48) as $\hat{\delta}_t = h(Q_t,\ \alpha_t,\ \Delta_t,\ \theta_{K,t-1})$ and $\hat{\delta}_t = \ell(Q_t,\ \alpha_t,\ \Delta_t,\ \theta_{K,t-1})$, respectively.

There are two cases to consider. In the first case, $\tilde{\delta}_{P,t} = 1 - \phi Q_t$ at the intersection of the two curves. In the second case, $\tilde{\delta}_{P,t} = \bar{\delta} - \Delta_t$ at the intersection.

In the first case, it can be shown that:

$$\frac{\partial h(Q_t,\ \alpha_t,\ \Delta_t,\ \theta_{K,t-1})}{\partial Q_t} = -\frac{\theta_{K,t-1}\phi\hat{\delta}_t(1-\tilde{\delta}_{P,t})}{\theta_{K,t-1}(\bar{\delta}+\Delta_t-\tilde{\delta}_{P,t})+\bar{\delta}+\Delta_t-\hat{\delta}_t}, \tag{68}$$

$$\frac{\partial \ell(Q_t,\ \alpha_t,\ \Delta_t,\ \theta_{K,t-1})}{\partial Q_t} = -\frac{\frac{\beta\alpha_t(1-\hat{\delta}_t)}{Q_t^2}+\frac{\theta_{K,t-1}\phi(1-\tilde{\delta}_{P,t})}{2\Delta_t}}{\beta\left(\frac{\alpha_t}{Q_t}+\frac{\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}\right)}, \tag{69}$$

which are always strictly negative. This result proves that the implicit functions h and ℓ exist by the implicit function theorem and that Equations (41) and (48) are downward-sloping on the $(Q_t, \hat{\delta}_t)$ plane.

It can be shown that $\frac{\partial \ell(Q_t,\, \alpha_t,\, \Delta_t,\, \theta_{K,t-1})}{\partial Q_t} - \frac{\partial h(Q_t,\, \alpha_t,\, \Delta_t,\, \theta_{K,t-1})}{\partial Q_t}$ has the same sign with:

$$-\left[\frac{\beta\alpha_t(1-\hat{\delta}_t)}{Q_t^2}+\frac{\theta_{K,t-1}\phi(1-\tilde{\delta}_{P,t})}{2\Delta_t}\right]\left[\theta_{K,t-1}(\bar{\delta}+\Delta_t-\tilde{\delta}_{P,t})+\bar{\delta}+\Delta_t-\hat{\delta}_t\right]$$
$$+\theta_{K,t-1}\phi\hat{\delta}_t(1-\tilde{\delta}_{P,t})\beta\left(\frac{\alpha_t}{Q_t}+\frac{\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}\right) \tag{70}$$

$$= -\frac{\beta\alpha_t(1-\hat{\delta}_t)}{Q_t^2}\left[\theta_{K,t-1}(\bar{\delta}+\Delta_t-\tilde{\delta}_{P,t})+\bar{\delta}+\Delta_t-\hat{\delta}_t\right]$$
$$-\theta_{K,t-1}\phi(1-\tilde{\delta}_{P,t})\left[\frac{\theta_{K,t-1}(\bar{\delta}+\Delta_t-\tilde{\delta}_{P,t})+\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}-\hat{\delta}_t\beta\left(\frac{\alpha_t}{Q_t}+\frac{\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}\right)\right]. \tag{71}$$

At the intersection of Equations (41) and (48), it holds that:

$$
\begin{aligned}
&\frac{\theta_{K,t-1}(\bar{\delta}+\Delta_t-\tilde{\delta}_{P,t})+\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}-\hat{\delta}_t\beta\left(\frac{\alpha_t}{Q_t}+\frac{\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}\right)\\
=&\frac{\theta_{K,t-1}(\bar{\delta}+\Delta_t-\tilde{\delta}_{P,t})+\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}\\
&-\frac{\hat{\delta}_t}{1-\hat{\delta}_t}\left[\theta_{K,t-1}\int_{\tilde{\delta}_{P,t}}^{\bar{\delta}+\Delta_t}\frac{1-\delta}{2\Delta_t}\,d\delta+\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta_t}\frac{1-\delta}{2\Delta_t}\,d\delta+(1-\beta)\int_{\bar{\delta}-\Delta_t}^{\hat{\delta}_t}\frac{1-\delta}{2\Delta_t}\,d\delta\right]\\
=&\left[\frac{\theta_{K,t-1}(\bar{\delta}+\Delta_t-\tilde{\delta}_{P,t})+\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}\right]\left(1-\hat{\delta}_t\right)-\frac{\hat{\delta}_t(1-\beta)}{1-\hat{\delta}_t}\int_{\bar{\delta}-\Delta_t}^{\hat{\delta}_t}\frac{1-\delta}{2\Delta_t}\,d\delta. \qquad (72)
\end{aligned}
$$

The first equality is obtained by substituting Equation (48), and the second equality is obtained by substituting Equation (41). Note that:

$$\frac{\hat{\delta}_t}{1-\hat{\delta}_t}\leq\frac{\bar{\delta}+\Delta_t}{1-(\bar{\delta}+\Delta_t)}, \qquad (73)$$

$$\int_{\bar{\delta}-\Delta_t}^{\hat{\delta}_t}\frac{1-\delta}{2\Delta_t}\,d\delta\leq 1-\bar{\delta}, \qquad (74)$$

by $\hat{\delta}_t \leq \bar{\delta}+\Delta_t$. Thus, the last term on the last line of Equation (72) goes to 0 as β goes to 1. This result proves that $\frac{\partial \ell(Q_t,\,\alpha_t,\,\Delta_t,\,\theta_{K,t-1})}{\partial Q_t} < \frac{\partial h(Q_t,\,\alpha_t,\,\Delta_t,\,\theta_{K,t-1})}{\partial Q_t}$ at the intersection of Equations (41) and (48) for the first case (i.e., $\tilde{\delta}_{P,t} = 1 - Q_t\phi$), if β is sufficiently close to 1.

For the second case (i.e., $\tilde{\delta}_{P,t} = \bar{\delta} - \Delta_t$), Equation (41) implies that $\hat{\delta}_t$ is constant. Also, it can be shown that:

$$\frac{\partial \ell(Q_t,\ \alpha_t,\ \Delta_t)}{\partial Q_t} = -\frac{\frac{\alpha_t(1-\hat{\delta}_t)}{Q_t^2}}{\frac{\alpha_t}{Q_t}+\frac{\bar{\delta}+\Delta_t-\hat{\delta}_t}{2\Delta_t}} < 0. \qquad (75)$$

Thus, the implicit functions h and ℓ exist, and Equation (48) has a steeper slope than Equation (41) on the $(Q_t, \hat{\delta}_t)$ plane.

C Proof of Lemma 4.2

Take the partial derivative:

$$\frac{\partial g(\hat{\delta}_t,\ Q_t,\ \alpha_t,\ \Delta_t,\ \theta_{K,t-1})}{\partial \Delta_t} = \frac{Q_t}{1-\hat{\delta}_t}\left\{ \frac{-\theta_{K,t-1}[1-(\bar{\delta}-\Delta_t)]}{2\Delta_t} + \theta_{K,t-1}\int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{1-\delta}{2(\Delta_t)^2}\, d\delta \right.$$
$$\left. -\beta\left[\frac{1-(\bar{\delta}-\Delta_t)}{2\Delta_t} - \int_{\bar{\delta}-\Delta_t}^{\hat{\delta}_t} \frac{1-\delta}{2(\Delta_t)^2}\, d\delta + \frac{1-\hat{\delta}_t}{2\Delta_t} - (1-\hat{\delta}_t)\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta_t} \frac{1}{2(\Delta_t)^2}\, d\delta\right]\right\}. \quad (76)$$

It can be shown that:

$$\frac{-[1-(\bar{\delta}-\Delta_t)]}{2\Delta_t} + \int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{1-\delta}{2(\Delta_t)^2}\, d\delta \begin{cases} < 0, & \text{if } \tilde{\delta}_{P,t} \le \bar{\delta}, \\ > 0, & \text{if } \tilde{\delta}_{P,t} \text{ is sufficiently close to } \bar{\delta}+\Delta_t, \end{cases} \quad (77)$$

given $\Delta_t \in (0, 1-\bar{\delta})$ and $\bar{\delta} \in (0,1)$.

Also, it can be shown that:

$$-\left[1-(\bar{\delta}-\Delta_t) - \int_{\bar{\delta}-\Delta_t}^{\hat{\delta}_t} \frac{1-\delta}{\Delta_t}\, d\delta + 1-\hat{\delta}_t - (1-\hat{\delta}_t)\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta_t} \frac{1}{\Delta_t}\, d\delta\right] \le 0, \quad (78)$$

given $\hat{\delta}_t \in [\bar{\delta}, \bar{\delta}+\Delta_t]$ by Equation (41), where equality holds if and only if $\hat{\delta}_t = \bar{\delta}+\Delta_t$. Note that Equation (41) implies that $\hat{\delta}_t$ is sufficiently close to $\bar{\delta}+\Delta_t$ if $\tilde{\delta}_{P,t}$ is sufficiently close to $\bar{\delta}+\Delta_t$. Thus, substituting Inequalities (77) and (78) in Equation (76) proves the proposition.

D Comparative statics in the model without a banking sector

Equations (30), (31) and (41) indicate that the distribution of net-worth among agents determines the value of $\hat{\delta}_t$. Figure 6 highlights this point by showing comparative statics around the benchmark parameter values specified in Section 5.2.

To simplify the parameter space of the model, suppose that the arrival of the opportunity to produce new capital is i.i.d. for each agent, i.e., $\rho_U = 1 - \rho_P$. The figure illustrates that the steady state value of $\hat{\delta}_t$ depends on the value of ρ_P, given that $\rho_U = 1 - \rho_P$. Note that the value of ρ_P determines the fraction of the productive in the economy and thus the distribution of net-worth between the productive and the unproductive. The figure shows that the change in $\hat{\delta}_t$ is non-monotonic.

There are two effects behind the determination of $\hat{\delta}_t$. First, $\hat{\delta}_t$ depends on the productive's share of supply of used capital in the market, since the productive sells better used capital than the unproductive, as implied by $\tilde{\delta}_{P,t} < \tilde{\delta}_{U,t}$. Thus, a drop in ρ_P increases $\hat{\delta}_t$, since the distribution of supply of used capital among agents is proportional to the distribution of net-worth among agents, as shown by Equations (30) and (31). Second, the value of Q_t increases in the aggregate net-worth of the unproductive, since the unproductive are the buyers of used capital and a rise in their aggregate net-worth increases the aggregate spending on used capital, as implied by Equation (31). If $\tilde{\delta}_{P,t} = 1 - \phi Q_t$, then an increase in Q_t in turn lowers $\tilde{\delta}_{P,t}$ by inducing the productive to sell better used capital, which reduces the value of $\hat{\delta}_t$ in the market. In the second effect, a drop in ρ_P lowers $\hat{\delta}_t$.

These two effects work in each other's opposite direction. In Figure 6, a drop in ρ_P lowers $\hat{\delta}_t$ through the second effect until $\tilde{\delta}_{P,t}$ hits the lower bound, $\bar{\delta} - \Delta_t$. After this point, the second effect disappears, and a drop in ρ_P increases $\hat{\delta}_t$ through the first effect.

E Proof of Proposition 5.1

First, prove Inequality (53). It is obvious that Inequality (53) holds when $\tilde{\delta}_{P,t} = \bar{\delta} - \Delta$ or $\tilde{\delta}_{P,t} = \bar{\delta} + \Delta$, given $\hat{\delta}_t > \bar{\delta}$ and Inequality (15). When $\tilde{\delta}_{P,t} = 1 - \phi Q_t$:

$$\frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} - \left(\int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{P,t}} \frac{(1-\delta)}{\phi \cdot 2\Delta_t} \, d\delta + \int_{\tilde{\delta}_{P,t}}^{\bar{\delta}+\Delta_t} \frac{Q_t}{2\Delta_t} \, d\delta \right)$$
$$= \frac{1}{\phi} \left\{ \frac{x(1-\bar{\delta})}{1-\hat{\delta}_t} - \frac{1 - x - y - \frac{(1-x)^2}{2} + \frac{y^2}{2}}{z-y} - \frac{x[z-(1-x)]}{z-y} \right\}, \quad (79)$$

where

$$x \equiv \phi Q_t, \quad y \equiv \bar{\delta} + \Delta, \quad z \equiv \bar{\delta} - \Delta. \quad (80)$$

Given the value of $\hat{\delta}_t$, the right-hand side of Equation (79) can be rewritten as a quadratic function of x. Note that $x \in [1-z, 1-y]$ by the definition of $\tilde{\delta}_{P,t}$ given by Equation (23). Since the coefficient of x^2 is negative, the right-hand side takes the minimum value for $x \in [1-z, 1-y]$ when $x = 1-z$ or $x = 1-y$. It can be shown that the right-hand side is positive in either case, given $\hat{\delta}_t > \bar{\delta}$ and Inequality (15).

Next, prove Inequality (54). Note that, by Equation (23), $\tilde{\delta}_{U,t} = \hat{\delta}_t$ if $\lambda_{U,t} = Q_t(1-\hat{\delta}_t)^{-1}$. Given $\hat{\delta}_t > \bar{\delta}$, if $\lambda_{U,t} = Q_t(1-\hat{\delta}_t)^{-1}$, then:

$$\frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} - \left(\int_{\bar{\delta}-\Delta_t}^{\tilde{\delta}_{U,t}} \frac{\lambda_{U,t}(1-\delta)}{2\Delta_t} \, d\delta + \int_{\tilde{\delta}_{U,t}}^{\bar{\delta}+\Delta_t} \frac{Q_t}{2\Delta_t} \, d\delta \right)$$
$$= \frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} - \left(\int_{\bar{\delta}-\Delta_t}^{\hat{\delta}_t} \frac{Q_t(1-\delta)}{(1-\hat{\delta}_t)2\Delta_t} \, d\delta + \int_{\hat{\delta}_t}^{\bar{\delta}+\Delta_t} \frac{Q_t}{2\Delta_t} \, d\delta \right) > \frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t} - Q_t > 0. \quad (81)$$

F A numerical solution method for the equilibrium dynamics of the model with a banking sector

The dynamic equilibrium is approximated by the following projection method:

Step 0. It can be shown that the dynamic equilibrium in each period is homogeneous of degree 1 with respect to $K_{P,t-1}$, $K_{U,t-1}$ and $K_{B,t-1}$. Set grid points on the state space for $K_{P,t-1}$, $K_{U,t-1}$ and the shock parameter (α_t or Δ_t). The value of $K_{B,t-1}$ is set to $1 - K_{P,t-1} - K_{U,t-1}$ on each grid point. Guess the equilibrium values of endogenous variables on each grid point, including $\bar{\omega}_{t+1}$ and $\underline{\omega}_{t+1}$. Call this correspondence between state variables and endogenous variables as a 'candidate array'.

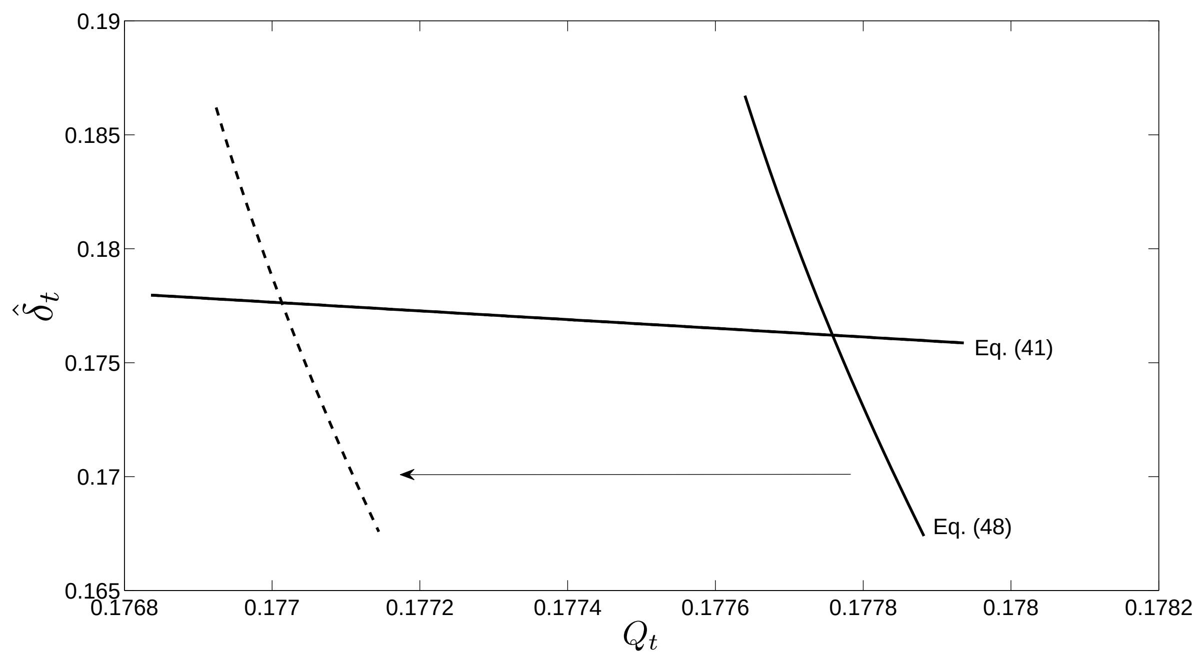
Step 1. Suppose the candidate array returns equilibrium values in the next period for each set of $K_{P,t}$, $K_{U,t}$ and the shock parameter. The equilibrium values on a point between the grid points in the state space are approximated by linear interpolation. Given this, derive the candidate array for the current period through the aggregate equilibrium conditions.

Step 2. Compare the candidate arrays for the current period and for the next period. If the ratio of each element between the two arrays becomes sufficiently close to 1, then take the candidate array as an equilibrium correspondence. Otherwise, update the candidate array for the next period by a linear combination of the two arrays and go back to Step 1.

In the numerical examples in this paper, I set grid points on the ± 5% range of the deterministic steady state values of $K_{P,t-1}$ and $K_{U,t-1}$. The number of grid points are 20 for these endogenous state variables. Note that the shock parameter only takes two values by assumption. The convergence criterion in Step 2 is 1e-03. In updating the candidate array in Step 2, the weight on the candidate array for the current period is 0.001. The initial guess in Step 0 is obtained through homotopy from the case where deterministic steady state values for each value of the shock parameter are a successful initial guess of the candidate array that leads to convergence.

The equilibrium conditions are checked for each element of the converged candidate array. For each figure, random simulations of the dynamics for 5000 periods confirm that the equilibrium dynamics move within the grid points that satisfy the equilibrium conditions.

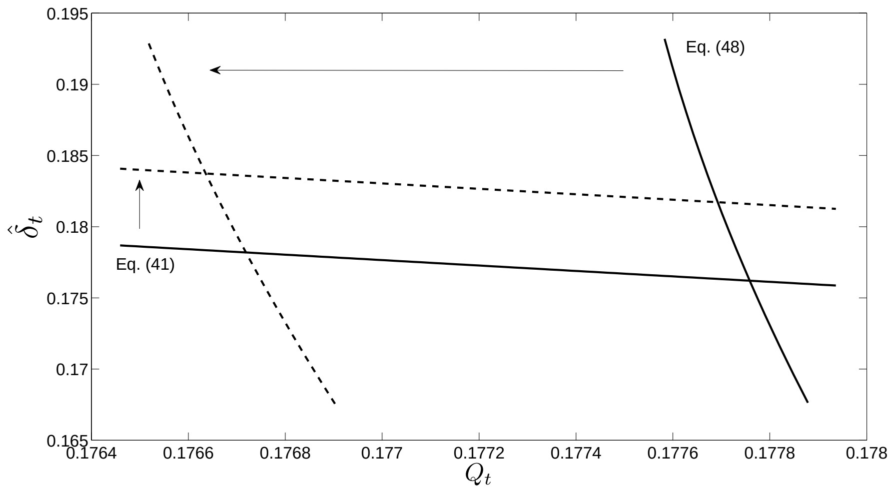
Figure 1: Dynamic equilibrium without the bank: the effect of a decline in α_t



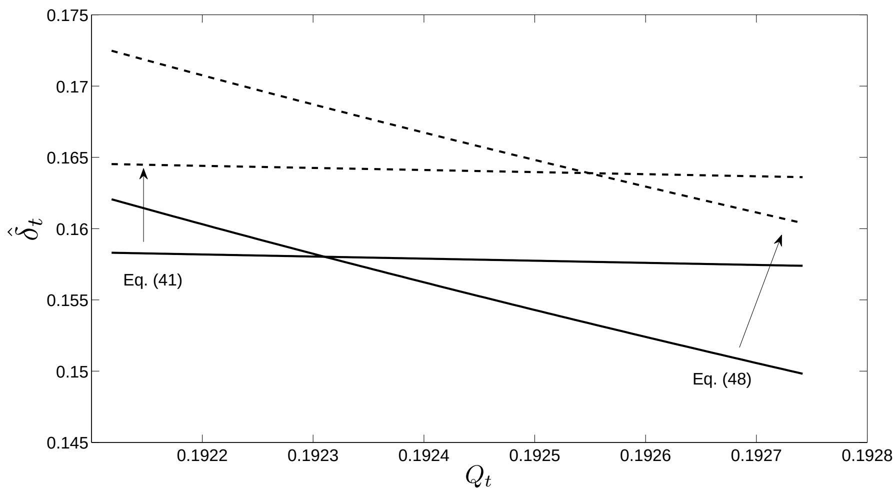
Notes: For all the curves in the figure, $K_{P,t-1}$ and $K_{U,t-1}$ take the deterministic steady state values. Parameter values used for deriving the deterministic steady state are $(\bar{\delta}, \phi, \beta, \zeta, \rho_P, \rho_U) = (0.1, 4.75, 0.99, 0.02, 0.45, 0.55)$, $\bar{\alpha} = \underline{\alpha} = 0.03$, and $\bar{\Delta} = \underline{\Delta} = 0.09$. The solid lines are Equations (41) and (48) with $\alpha_t = 0.03$ and the dashed lines are these equations with $\alpha_t = 0.027$.

Figure 2: Dynamic equilibrium without the bank: the effect of an increase in Δ_t

(a)

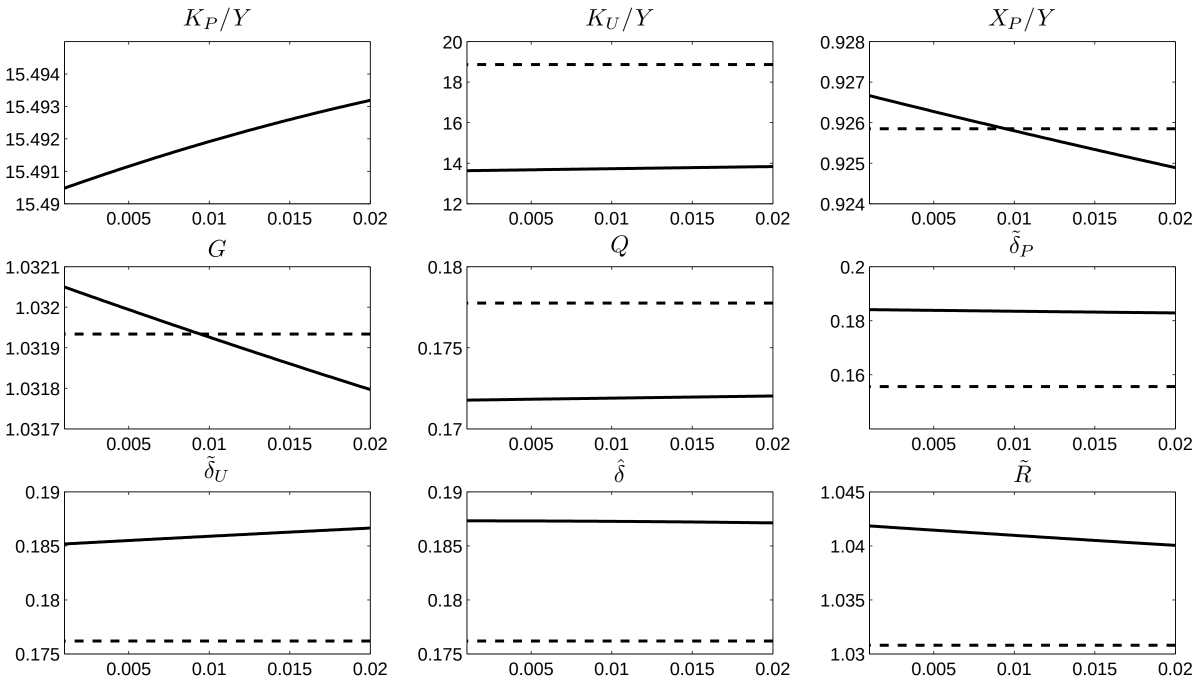


(b)



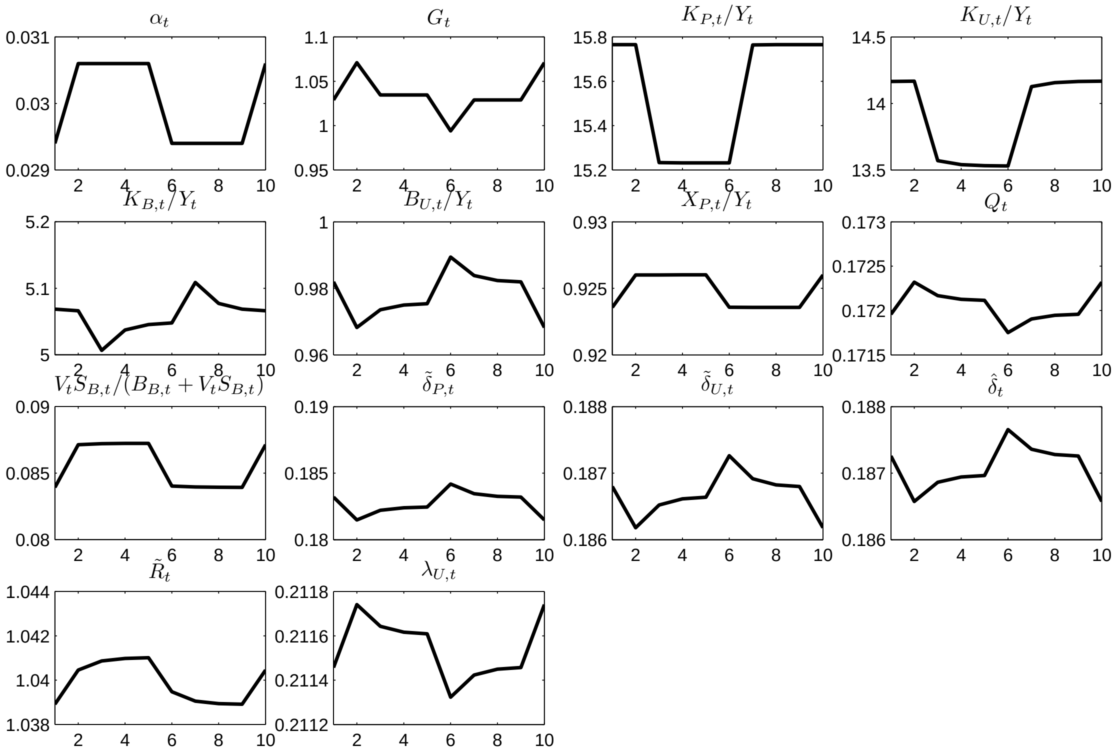
Notes: $K_{P,t-1}$ and $K_{U,t-1}$ take the deterministic steady state values. Parameter values used for deriving the deterministic steady state are: $(\bar{\delta},\, \phi,\, \beta,\, \zeta) = (0.1,\, 4.75,\, 0.99,\, 0.02)$, $\bar{\alpha} = \underline{\alpha} = 0.03$ and $\bar{\Delta} = \underline{\Delta} = 0.09$ for both panels; $(\rho_P,\, \rho_U) = (0.45,\, 0.55)$ for the top panel; and $(\rho_P,\, \rho_U) = (0.2,\, 0.8)$ for the bottom panel. In each panel, the solid lines are Equations (41) and (48) with $\Delta_t = 0.09$ and the dashed lines are these equations with $\Delta_t = 0.099$.

Figure 3: Comparative statics: with and without the bank



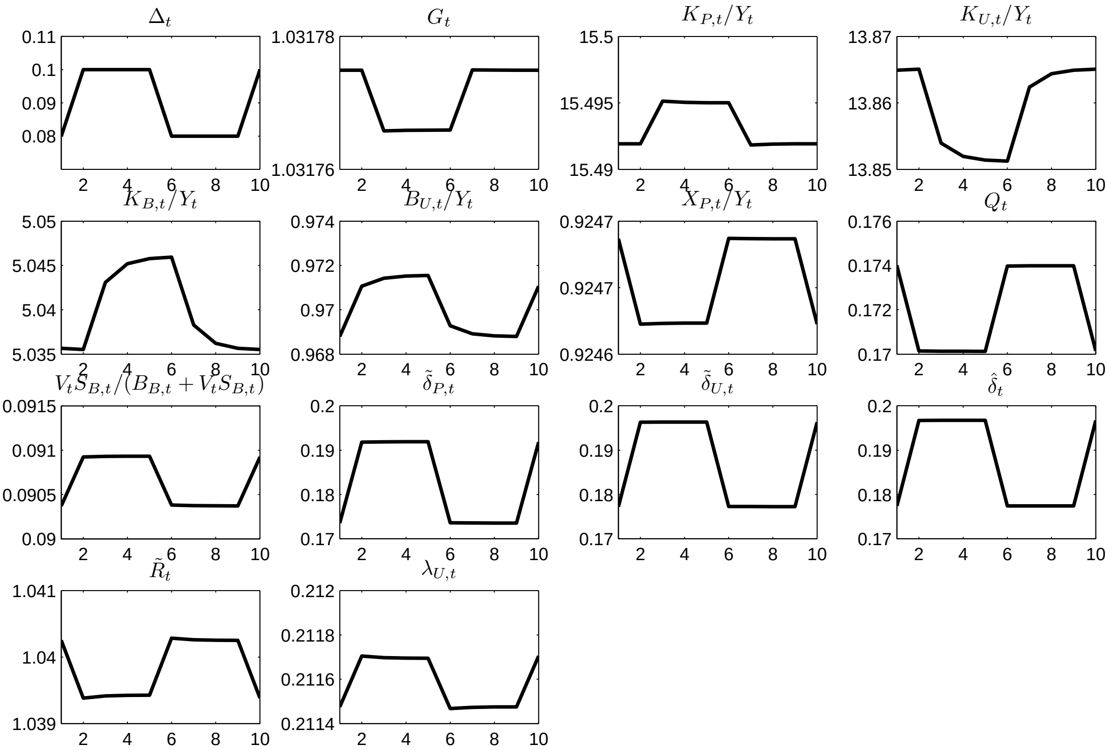
Notes: The horizontal axis is the value of ζ. The solid lines represent the model with the bank and the dashed lines represent the model without the bank. Parameter values are $(\bar{\delta},\ \phi,\ \beta,\ \rho_P,\ \rho_U) = (0.1,\ 4.75,\ 0.99,\ 0.45,\ 0.55)$, $\bar{\alpha} = \underline{\alpha} = 0.03$ and $\bar{\Delta} = \underline{\Delta} = 0.09$. Each panel of the figure shows the value of the variable in the title on the balanced growth path.

Figure 4: Dynamic equilibrium with the bank: business cycles driven by α_t



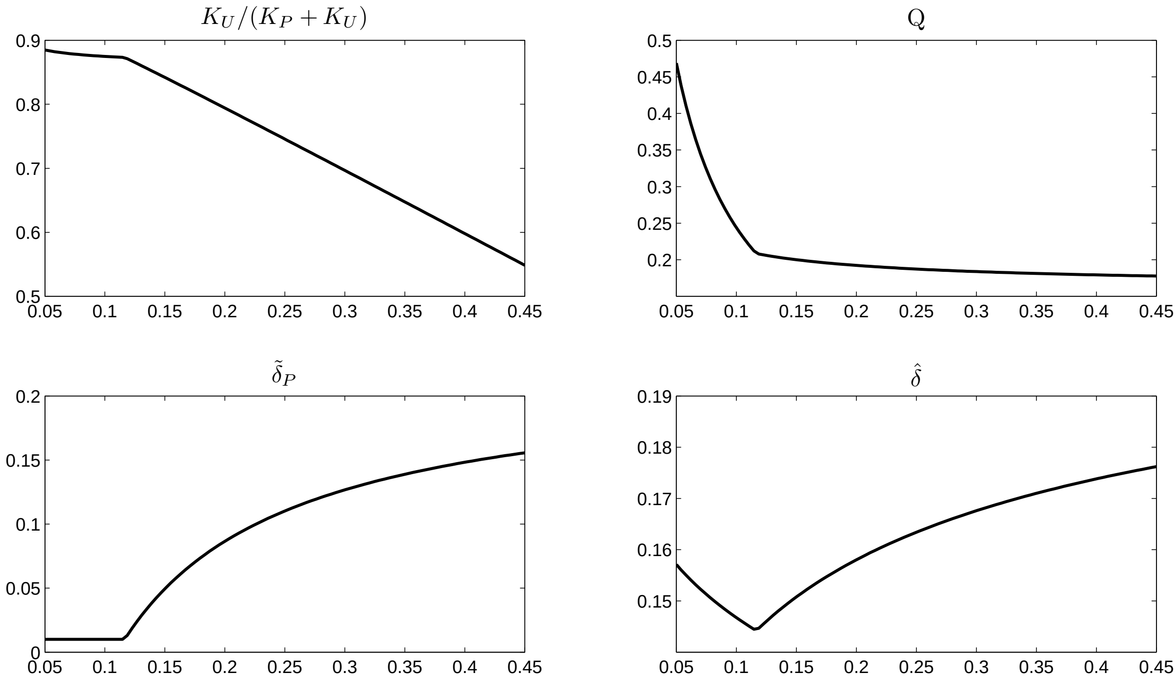
Notes: Parameter values are $(\bar{\delta},\ \phi,\ \beta,\ \zeta,\ \rho_P,\ \rho_U) = (0.1,\ 4.75,\ 0.99,\ 0.02,\ 0.45,\ 0.55)$, $(\bar{\alpha},\ \underline{\alpha}) = (0.0306,\ 0.0294)$, $\eta_{\bar{\alpha}} = \eta_{\underline{\alpha}} = 0.75$, and $\bar{\Delta} = \underline{\Delta} = 0.09$. The figure shows a sample path when α_t keeps changing its value every 4 periods for a sufficiently long time. $K^o_{U,t} = 0$ for all the periods.

Figure 5: Dynamic equilibrium with the bank: business cycles driven by Δ_t



Notes: Parameter values are $(\bar{\delta}, \phi, \beta, \zeta, \rho_P, \rho_U) = (0.1, 4.75, 0.99, 0.02, 0.45, 0.55)$, $\bar{\alpha} = \underline{\alpha} = 0.03$, $(\bar{\Delta}, \underline{\Delta})$ $= (0.1, 0.08)$, and $\eta_{\bar{\Delta}} = \eta_{\underline{\Delta}} = 0.75$. The figure shows a sample path when Δ_t keeps changing its value every 4 periods for a sufficiently long time. $K^o_{U,t} = 0$ for all the periods.

Figure 6: Comparative statics without the bank: net-worth distribution and illiquidity of capital (for Appendix D)



Notes: The horizontal line of each panel is ρ_P, given $\rho_P = 1 - \rho_U$. Parameter values are ($\bar{\delta}$, ϕ, β ζ) = (0.1, 4.75, 0.99, 0.02), $\bar{\alpha} = \underline{\alpha} = 0.03$, and $\bar{\Delta} = \underline{\Delta} = 0.09$. Each panel of the figure shows the value of the variable in the title on the balanced growth path.