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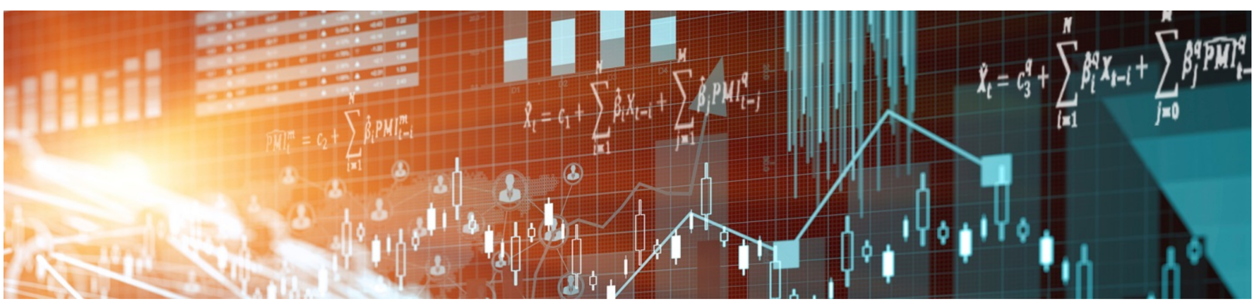
# Ambiguity, Nominal Bond Yields and Real Bond Yields



by Guihai Zhao

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# Ambiguity, Nominal Bond Yields and Real Bond Yields

by

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## Abstract

Equilibrium bond-pricing models rely on inflation being bad news for future growth to generate upward-sloping nominal yield curves. We develop a model that can generate upward-sloping nominal and real yield curves by instead using ambiguity about inflation and growth. Ambiguity can help resolve the puzzling fact that upward-sloping yield curves have persisted despite positive inflation shocks changing from negative to positive news about growth in the last twenty years. Investors make decisions using worst-case beliefs, under which the expectations hypothesis roughly holds. However, inflation and growth evolve over time under the true distribution, and this difference makes excess returns on long-term bonds predictable. The model is also consistent with the recent empirical findings on the term structure of equity returns.



*Bank topics: Asset pricing; Financial markets; Interest rates*
*JEL codes: G00, G12, E43*



## Résumé

Les modèles d'équilibre servant à l'évaluation des obligations reposent sur l'hypothèse que l'inflation a pour effet de freiner la croissance future, ce qui se traduit par une courbe des rendements nominaux ascendante. Nous concevons un modèle pouvant générer des courbes ascendantes pour les rendements nominaux et réels, en tenant compte d'un certain degré d'ambiguïté à l'égard de l'inflation et de la croissance. Cette ambiguïté peut contribuer à expliquer l'étonnante persistance des courbes de rendement ascendantes en dépit des chocs d'inflation positifs, qui ont eu des répercussions positives plutôt que négatives sur la croissance au cours des vingt dernières années. Les investisseurs fondent leurs décisions sur l'hypothèse que le pire des scénarios se réalisera, et dans ce cadre, l'hypothèse relative aux attentes se vérifie globalement. Or, l'évolution de l'inflation et de la croissance au fil du temps ne reflète pas l'hypothèse du pire scénario, et cette différence fait en sorte qu'il est possible de prévoir les excédents de rendement des obligations à long terme. Les résultats obtenus à l'aide de notre modèle concordent également avec les analyses empiriques récentes de la structure par terme des rendements des actions.



*Sujets : Évaluation des actifs; Marchés financiers; Taux d'intérêt*
*Codes JEL : G00, G12, E43*

**Non-Technical Summary**

This paper develops an equilibrium asset pricing model to solve three puzzles in finance. First, to generate an upward-sloping nominal yield curve, equilibrium bond-pricing models rely on the assumption that inflation is bad news for future growth. However, today inflation is considered good news for future growth. Second, despite strong evidence of realized excess bond return predictability, the expectations hypothesis roughly holds under the subjective expectations from the survey. Finally, the term structure of Treasury inflation-protected securities is upward sloping in the U.S.

The three puzzles are tightly connected, and the challenge is to explain them simultaneously. Departing from the rational expectation hypothesis, we assume that the investor is ambiguity averse and evaluates future prospects under a worst-case scenario. The term structure of ambiguity for inflation is upward sloping before the late 1990s, and slopes downward afterwards, while the ambiguity yield curve for real output growth is always downward sloping. The ambiguity yields are linked with bond yields and equity yields through the recursive multiple priors preference in equilibrium.

For both subperiods, the worst-case distribution for output growth is the lower bound of the set of alternative mean growth rates, which is upward sloping because of the downward-sloping output forecast dispersion. Thus the real bond yield is always upward sloping. Before the late 1990s, when inflation expectation is negatively associated with the worst-case growth expectation, ambiguity averse investors pick the upper bound from the set of alternative mean inflation scenarios, which is upward sloping. This generates an upward-sloping nominal yield curve. During the second subperiod, inflation expectation is positively associated with the worst-case growth expectation, and the worst-case mean inflation becomes the lower bound. However, at the same time, the inflation forecast dispersion turns to be downward sloping, which again implies an upward-sloping mean inflation in equilibrium. Therefore the model generates upward-sloping nominal yield curves in both subperiods, but with a different mechanism.

## 1. Introduction

To be consistent with the fact that the nominal yield curves are upward sloping, equilibrium bond-pricing models rely on inflation as bad news for future growth and the assumption that agents prefer early resolution of uncertainty; see, for example, Piazzesi and Schneider (2007) (henceforth PS 2007) and Bansal and Shaliastovich (2013). The intuition is that a positive surprise to inflation lowers future consumption growth, and at the same time, decreases the real payoff of long-term nominal bonds. Therefore, long-term nominal bonds are risky and command a term spread over short-term bonds. However, in the current macroeconomic environment where inflation is good news for future growth, these models also imply a downward-sloping nominal yield curve, which is in contrast to the fact that in the data the nominal yield curve continues to slope up after the late 1990s.[2] This paper provides an alternative approach to understand upward-sloping nominal yield curves in both environments.

An important related fact is excess bond return predictability. Against the expectations hypothesis, Fama and Bliss (1987), Campbell and Shiller (1991), Dai and Singleton (2002), and Cochrane and Piazzesi (2005) provide evidence for bond return predictability using yield spreads and forward rates as predictors. Others, however, show that the failure of the expectations hypothesis is due to expectational errors (Froot (1989); Piazzesi et al. (2015)). We will show that these results can be reconciled if investors have equilibrium subjective beliefs that are different from the reference distribution.

From the perspective of equilibrium asset pricing models, another puzzling fact is that the term structure of Treasury inflation-protected securities (TIPS) is upward sloping in the U.S. In the twenty-year history of TIPS data, the observed slope has never been significantly negative. Campbell (1986) shows that real bonds have a negative real term premium if consumption growth follows a persistent process. While it has been difficult

[2]Recent developments in the bond market literature have shown that the correlation between consumption growth and inflation has switched from negative to positive after the late 1990s, which can explain the changes in correlation between U.S. Treasury bond returns and stock returns. See, for example, Burkhardt and Hasseltoft (2012); David and Veronesi (2013); Campbell et al. (2016); Song (2017).

to account for the nominal bond yield curve and bond return predictability, it is much harder for an equilibrium model of bond pricing to also capture real bond yields. In fact, except for Wachter (2006), the previously mentioned models generate a downward-sloping real yield curve. Finally, the recent empirical findings on the term structure of equity returns pose some serious challenges to equilibrium models.[3]

This paper develops a consumption-based asset pricing model that helps to explain the preceding features in the data by positing that investors have limited information about the stochastic environment and hence face both risk and ambiguity. Risk refers to the situation where there is a probability law to guide choice. However, there is incomplete confidence that any single distribution accurately describes the environment, and ambiguity refers to the case where there is uncertainty about the distribution. Specifically, we assume that there is ambiguity about both real growth and inflation distribution. Using forecast dispersion as an empirical measure for the size of ambiguity (or confidence), we find that, before the late 1990s, the size of ambiguity for long horizon inflation is bigger than those for short horizons, and the term structure of ambiguity is reversed afterwards. However, the term structure of ambiguity for real output growth is always downward sloping. In equilibrium, ambiguity averse agents evaluate future prospects under the worst-case measure. Given the term structure of ambiguity for inflation and real growth, we show that, in equilibrium, the worst-case growth and inflation expectations are upward sloping for both subperiods, which generates upward-sloping nominal and real yield curves in both environments.

Departing from the rational expectation model, we assume that investors are ambiguity averse and have recursive multiple priors (or maxmin) preferences with a constant relative risk aversion (CRRA) utility (Epstein and Schneider (2003)). Investors in this economy have in mind a benchmark or reference measure of the economy's dynamics that represents the best point estimate of the stochastic process. As in PS 2007, under the reference benchmark, real growth and inflation are described by a state space model.

[3]See, for example, Van Binsbergen and Koijen (2017) for a survey.

However, investors are concerned that the reference measure is misspecified and believe that the true measure is actually within a set of alternative measures that are statistically close to the reference distribution.

The set of alternative measures for real growth/inflation is generated by a set of different mean real growth/inflation rates around its reference mean value. We use the Blue Chip Financial Forecast (BCFF) survey to characterize the properties of ambiguity yields for U.S. real output growth and the consumer price index (CPI) from 1985 to 2017. Motivated by the fact that inflation forecast dispersion has switched from upward sloping to downward sloping after the late 1990s, we model inflation ambiguity as a random walk with positive drift in the first subperiod and with negative drift in the second subperiod. Given that real output growth forecast dispersion has been consistently downward sloping, we assume that ambiguity about real growth is a random walk with negative drift in both periods. We assume an unexpected discrete regime shift mainly due to changes in inflation patterns and monetary policy with the first subperiod as the inflation fighting period of Volcker and Greenspan and the second subperiod as the recent period of low inflation and increased central bank transparency.[4] One possible interpretation for the observed change in term structure of forecast dispersion is that, as argued by Goodfriend and King (2005), "inflation scares" were created during the monetary policy experimentation of the late 1970s and early 1980s, and investors were not sure about future inflation scenarios until inflation was fully under control after the late 1990s. Currently, investors have less ambiguity regarding longer horizon inflation due to a clear understanding of inflation targeting and the low inflation environment.

In equilibrium, the values of bonds and dividend strips can be solved as functions of the ambiguity processes. For the whole period, ambiguity averse agents make decisions using the lower bound of the set of alternative mean output growth–the worst-case measure–which is upward sloping because of the downward-sloping dispersion yields for output forecasts. Thus the real bond yield curve is always upward sloping. During the

[4]See, for example, Campbell et al. (2014) and Zhao (2017) for a similar regime break. The results are robust to different regime break points.

first subperiod, when inflation expectation is negatively associated with the worst-case expected real output growth, the worst-case mean inflation is the upper bound, which is upward sloping because the dispersion is bigger for a longer horizon. This implies that investors' subjective nominal short rate expectation is upward sloping, which generates an upward-sloping nominal yield curve. During the second subperiod, inflation expectation becomes positively associated with the worst-case expected real output growth, and the worst-case mean inflation becomes the lower bound. However, at the same time, the inflation forecast dispersion turns to be downward sloping, which again implies an upward-sloping mean inflation in equilibrium. Therefore the model generates upward-sloping nominal yield curves in both subperiods, but with a different mechanism. The model-implied bond yield volatility is also consistent with data across periods.

Many studies have documented that excess returns on long-term bonds are predictable. However, using survey expectations as subjective beliefs, a small literature argues that the failure of the expectations hypothesis is due to expectational errors. For example, Piazzesi et al. (2015) show that the expected excess returns on long-term bonds consist of two parts: the expected subjective bond premium and the difference between subjective and statistical future interest rate expectations, and they find the second part is significant. In our model, yields for long-term bonds are roughly equal to the average of expected future short rates under the equilibrium worst-case belief. Thus, consistent with Froot (1989) and Piazzesi et al. (2015), the expectations hypothesis roughly holds under the subjective equilibrium belief. However, one part of the ambiguity (about long-run inflation or GDP growth expectations) does not materialize when the time arrives, thus there is no trend in the realized ambiguity process. Both this difference and current yield spreads/forward rates are driven by the trend components in the ambiguity process. Hence, consistent with the empirical evidence, the realized excess bond returns are predictable in the model.

Even though the model focuses primarily on bond yields, it has important implications for the term structure of dividend strips as well. The empirical findings on equity yields are different across countries. Using dividend future contracts for the S&P500,

Van Binsbergen and Koijen (2017) show that dividend future returns are slightly upward sloping and the volatility of equity yields is downward sloping, and the market returns are not significantly different from individual dividend spot returns. This model is consistent with these findings.

*Related literature*

This paper is closely related to some recent developments in equilibrium bond-pricing models. Using Epstein and Zin (1989) preferences, PS 2007 show that inflation as bad news for future consumption growth can generate an upward-sloping nominal yield curve. In a similar vein, Wachter (2006) generates upward-sloping nominal and real yield curves in an external habits model (Campbell and Cochrane (1999)), where innovations to consumption and inflation growth are negatively correlated. Taking inflation as bad news for future growth, Bansal and Shaliastovich (2013) show that a long-run risks model with time-varying volatility of expected consumption growth and inflation can account for bond return predictability. Ulrich (2013) argues that, even with log utility, ambiguity about trend inflation can help generate an upward-sloping term premium for nominal bonds if inflation shocks make the size of ambiguity bigger. While these studies argue that a single mechanism can explain the yield curve for the whole sample period (no regime switch), Song (2017) extends the long-run risks model of Bansal and Yaron (2004) by allowing a regime switch in the correlation between consumption growth and the inflation target. He finds that the U.S. economy entered a positive correlation regime following the late 1990s and has largely remained in that regime thereafter. Song (2017) argues that if agents evaluate long-term bonds using an unconditional probability of switching from a positive correlation regime to a negative one, the long-run risks model generates an upward-sloping nominal yield curve.[5] However, based on Malmendier and Nagel (2016),

[5] In the positive correlation regime of the current period, the *conditional* probability of switching back to a negative correlation regime is close to zero, while the *unconditional* probability is about 2/3 because the economy has been in the negative correlation regime most periods before the late 1990s. Due to the downward-sloping real yield curve in the model, the model-implied nominal yield curve slope is only 1/3 of the data, even using the unconditional probability in Song (2017).

agents are more likely to use a conditional probability. This is because agents are more likely to use experiences rather than unconditional means.

This paper differs from these previous studies along some important dimensions. First, we provide an alternative understanding of the upward-sloping nominal yield curve for two environments where inflation can be bad or good news for future growth. Second, this paper provides a new mechanism to generate an upward-sloping real yield curve for both the pre- and post-2000s. Wachter (2006) is the only other paper we know of that can generate an upward-sloping real yield curve. We also show that, in this model, the ambiguity term premium that Ulrich (2013) uses to generate upward-sloping nominal bond yields is quantitatively very small. The upward-sloping feature for bond yields is mainly driven by the term structure of ambiguity. From the perspective of equilibrium models, this paper is the first effort, to our knowledge, to jointly understand upward-sloping real and nominal bond yield curves across different subperiods.

This paper is also related to a large empirical literature on excess bond return predictability (Fama and Bliss (1987); Campbell and Shiller (1991); Dai and Singleton (2002); Cochrane and Piazzesi (2005)), and a small empirical literature that argues the failure of the expectations hypothesis is due to expectational errors (Froot (1989); Piazzesi et al. (2015)). This is the first paper that provides a theoretical framework that is consistent with both of these findings.

This paper is related to a number of papers that have studied the implications of ambiguity and robustness for finance and macroeconomics (see the survey by Epstein and Schneider (2010) and the references therein). Ilut and Schneider (2014) show how time-varying ambiguity about productivity generates business cycle fluctuations. Using forecast dispersion data, Zhao (2017) shows that ambiguity about consumption growth is driven by past inflation and argues that bond risk changes are due to the time-varying impact of inflation on ambiguity. This paper contributes to the ambiguity literature by first showing a different term structure of ambiguity for inflation and output growth over two subperiods, and then using the recursive multiple-priors preference to link ambiguity yields with real and nominal bond yields and the equity yields.

The paper continues as follows. Section 2 outlines the model and solves it analytically. Section 3 discusses the results of the empirical analysis. Section 4 provides concluding comments.

## 2. The model

In a pure exchange economy, identical ambiguity averse investors maximize their utility over endowment/output processes. Output growth and inflation are given exogenously. Equilibrium prices adjust such that the agent is happy to consume the endowment.[6]

### *2.1. Economy dynamics*

Under reference measure $P$, output growth and inflation follow a state space model, while dividend growth is leveraged output growth:

$$
\begin{aligned}
\Delta g_{t+1} &= \mu_c + x_{c,t} + \sigma_c \varepsilon_{c,t+1} \\
\pi_{t+1} &= \mu_\pi + x_{\pi,t} + \sigma_\pi \varepsilon_{\pi,t+1} \\
x_{c,t+1} &= \rho_c x_{c,t} + \sigma_c^x \varepsilon_{c,t+1} + \sigma_{c\pi}^x \varepsilon_{\pi,t+1} \\
x_{\pi,t+1} &= \rho_\pi x_{\pi,t} + \sigma_\pi^x \varepsilon_{\pi,t+1} \\
\Delta d_{t+1} &= \zeta_d \Delta g_{t+1} + \mu_d + \sigma_d \varepsilon_{d,t+1}
\end{aligned}
\tag{1}
$$

where $\Delta g_{t+1}$ and $\Delta d_{t+1}$ are the growth rate of output and dividends respectively, and $\pi_t$ is inflation. The expected growth and inflation are denoted by $x_{c,t}$ and $x_{\pi,t}$. As argued in PS 2007, the state space representation for $z_{t+1} = (\Delta g_{t+1}, \pi_{t+1})^T$ does a good job in capturing the dynamics of inflation, especially the high order autocorrelations. For simplicity, we assume that the correlation between growth and inflation is captured by $\sigma_{c\pi}^x$. All shocks are i.i.d normal and orthogonal to each other.

[6] We use output growth as the endowment process because the non-durable good and service survey is not available in the BCFF. Using the Philadelphia Fed's Survey of Professional Forecasters (SPF), Zhao (2017) shows that the dispersion for consumption growth and output growth are highly correlated.

To model dividends and output separately, we follow Ju and Miao (2012), where the parameter $\zeta_d > 0$ can be interpreted as the leverage ratio on expected output growth, as in Abel (1999); together with the parameter $\sigma_d$, this allows us to calibrate the correlation of dividend growth with consumption growth. The parameter $\mu_d$ helps match the expected growth rate of dividends.

The above state space system for inflation and output growth represents the best point estimate from the data. However, investors are concerned that this reference measure is misspecified and that the true measure is actually within a set of alternative measures that are statistically close to the reference measure.

*2.2. Ambiguity about inflation and output growth*

The early ambiguity literature focuses on either the real economy, for example, ambiguity about consumption growth/TFP growth, or on the nominal side, for example, ambiguity about inflation. However, due to the very different patterns of the observed forecast dispersion for inflation and output growth, in this paper, we assume that investors are ambiguous about both inflation and output growth. The set of alternative measures is generated by a set of different mean output growth (inflation) rates around the reference mean value $\mu_c + x_{c,t}$ $(\mu_\pi + x_{\pi,t})$.[7] Specifically, under alternative measure $p^{\tilde{\mu}}$, output growth and inflation are as follows:

$$\begin{aligned} \Delta g_{t+1} &= \tilde{\mu}_{c,t} + x_{c,t} + \sigma_c \tilde{\varepsilon}_{c,t+1} \\ \pi_{t+1} &= \tilde{\mu}_{\pi,t} + x_{\pi,t} + \sigma_\pi \tilde{\varepsilon}_{\pi,t+1} \end{aligned} \tag{2}$$

where $\tilde{\mu}_{c,t} \in A_{c,t} = [\mu_c - a_{c,t}, \mu_c + a_{c,t}]$ and $\tilde{\mu}_{\pi,t} \in A_{\pi,t} = [\mu_\pi - a_{\pi,t}, \mu_\pi + a_{\pi,t}]$ with both $a_{c,t}$ and $a_{\pi,t}$ being positive. Each trajectory of $\tilde{\mu}_t$ will yield an alternative measure $p^{\tilde{\mu}}$ for the joint process. A larger $a_{c,t}(a_{\pi,t})$ implies that investors are less confident about the reference distribution. In the following section, we specify how ambiguity changes over

[7] One requirement for the alternative measures is that they must be equivalent to the reference measure $P$ (i.e., they put positive probabilities on the same events as $P$).

time and model the different term structure of ambiguity.

*2.3. Term structure of ambiguity*

To measure ambiguity empirically, we follow the literature and use the forecast dispersion from the BCFF survey.[8] As argued in Ilut and Schneider (2014), the reason is that investors sample experts' opinions and aggregate them when making decisions. Thus large disagreement among experts makes investors less confident in their probability assessments, which corresponds to a bigger size of ambiguity. We use BCFF forecast dispersion for GDP growth and CPI inflation from 1985 to 2017.[9] The BCFF survey contains forecasts for short-term and long-term horizons from the same participants, and the dispersion is calculated as the difference between the top 10 average and bottom 10 average of the individual forecasts in levels.

Figure 1 shows the one-quarter-ahead and six-years-ahead forecast dispersion for CPI inflation from 1985 to 2017. It is clear that six-years-ahead dispersion is bigger than one-quarter-ahead dispersion before the late 1990s, and the relationship is reversed afterwards. One possible interpretation is that, as argued by Goodfriend and King (2005), "inflation scares" were created during the monetary policy experimentation of the late 1970s and early 1980s, and investors were initially unsure about future inflation scenarios. Currently, investors have less ambiguity regarding longer horizon inflation due to a clear understanding of inflation targeting and the low inflation environment. Figure 2 plots the long and short horizon forecast dispersion for real GDP growth, and it suggests that long horizon dispersion is smaller than short horizon dispersion for most periods (except for a few periods around 1992). One reason for this may be due to the fact that investors understand that GDP growth is always the key mandate for the Federal Reserve Bank.

Table 1 shows quantitatively that the term structure of inflation forecast dispersion

[8] See, for example, Anderson, Ghysels, and Juergens (2009), Ilut and Schneider (2014), Drechsler (2013), and Zhao (2017).

[9] There are two reasons why we use the BCFF instead of other surveys such as the Philadelphia Fed's SPF. The first one is that the number of forecasters are more stable for the BCFF, which means the forecast dispersion is more accurate. The second is that the BCFF provides monthly survey results, which gives us more data points.

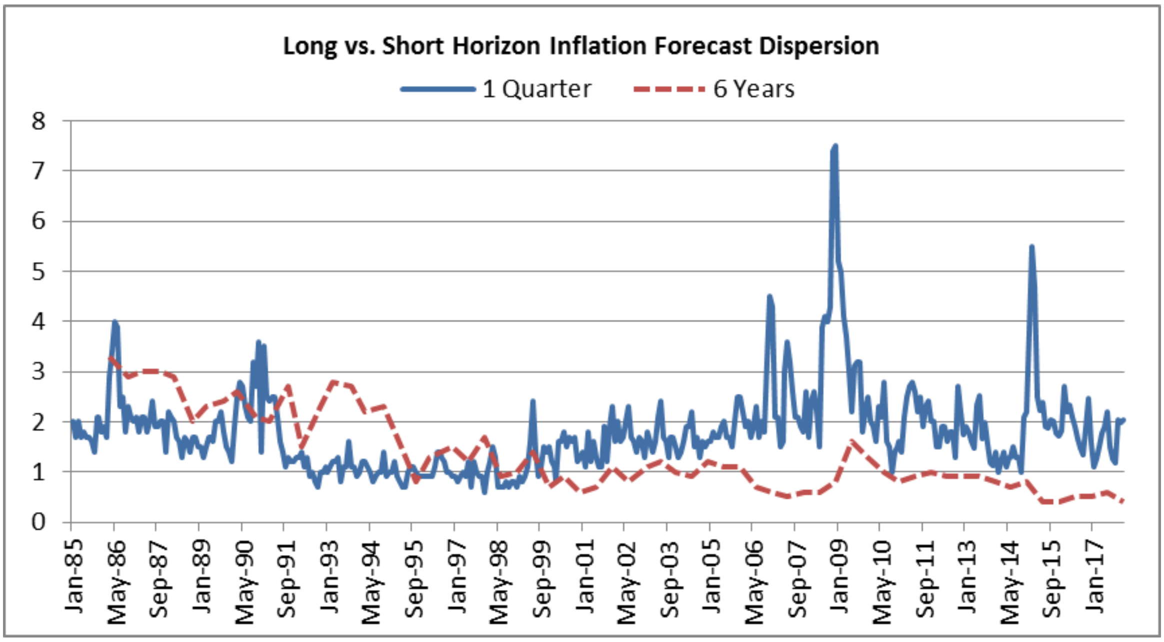


Figure 1: Term structure of ambiguity/dispersion for inflation
The dispersion is for one-quarter-ahead and six-years-ahead inflation forecasts from the BCFF from 1985 to 2017. One-quarter-ahead forecasts are monthly and six-years-ahead forecasts are semiannually.

has switched from upward sloping to downward sloping after the late 1990s. However, we still observe a significant amount of dispersion for even six-years-ahead inflation forecasts in the second subperiod. For real GDP growth, the term structure of forecast dispersion is consistently downward sloping across the two subperiods, and similar to inflation, we observe a significant amount of dispersion for six-years-ahead forecasts in both subperiods.

Motivated by the observed term structure of ambiguity, we model $a_{c,t}$ and $a_{\pi,t}$ as a random walk with drift that are specified in the following way:

$$
\begin{aligned}
a_{c,t+1} &= \mu_c^a + a_{c,t} + \sigma_{ac}\varepsilon_{ac,t+1} + \sigma_a^{ac}\varepsilon_{a,t+1} \\
a_{\pi,t+1} &= \mu_\pi^a + a_{\pi,t} + \sigma_a^{a\pi}\varepsilon_{a,t+1} \qquad (3)
\end{aligned}
$$

where $\mu_c^a$ and $\mu_\pi^a$ are the drift parameters, which can be positive or negative. Given the high correlation between inflation and GDP growth dispersion in the data, both $a_{c,t}$ and $a_{\pi,t}$ are driven by a common exogenous shock $\varepsilon_{a,t+1}$, where the coefficients $\sigma_a^{ac}$ and $\sigma_a^{a\pi}$ capture the correlation between them. $\varepsilon_{ac,t+1}$ is an $a_{c,t}$ specific shock that captures the

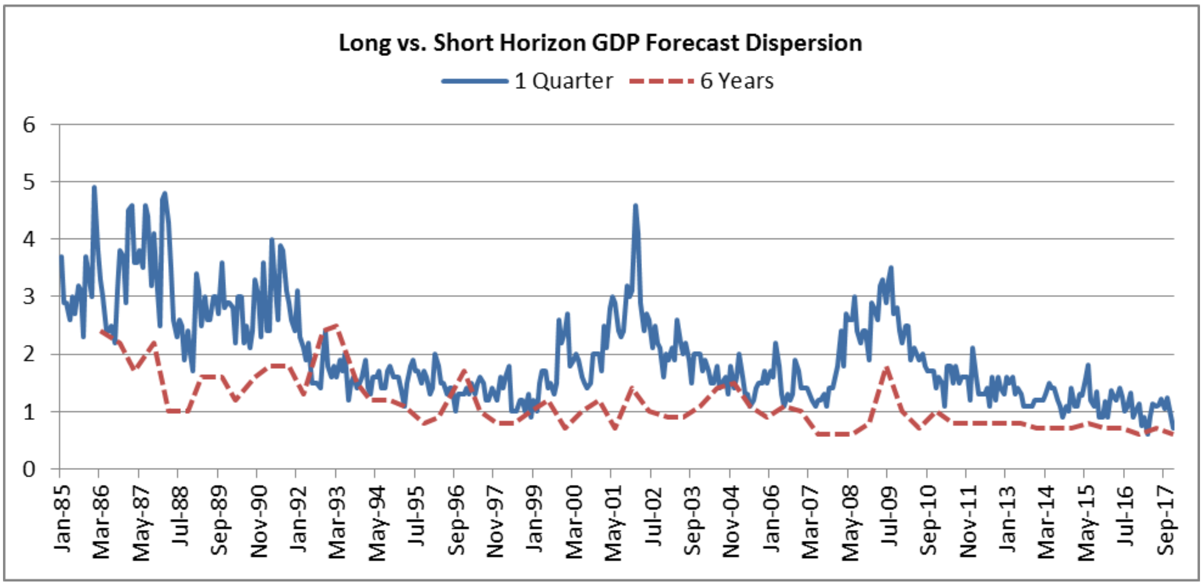


Figure 2: Term structure of ambiguity/dispersion for GDP

The dispersion is for one-quarter-ahead and six-years-ahead GDP forecasts from the BCFF from 1985 to 2017. One-quarter-ahead forecasts are monthly and six-years-ahead forecasts are semiannually.

| | 1985–1999 | 2000–2017 |
|---|---|---|
| Inflation_Disp_Q1 | 1.49 | 2.05 |
| Inflation_Disp_Q2 | 1.56 | 1.71 |
| Inflation_Disp_Q3 | 1.72 | 1.54 |
| Inflation_Disp_Q4 | 1.90 | 1.45 |
| Inflation_Disp_Q5 | 2.03 | 1.40 |
| Inflation_Disp_6Y | 2.04 | 0.83 |
| GDP_Disp_Q1 | 2.27 | 1.73 |
| GDP_Disp_Q2 | 2.55 | 1.78 |
| GDP_Disp_Q3 | 2.52 | 1.69 |
| GDP_Disp_Q4 | 2.38 | 1.55 |
| GDP_Disp_Q5 | 2.31 | 1.42 |
| GDP_Disp_6Y | 1.44 | 0.90 |

Table 1: Term structure of dispersion

Table 1 reports the term structure of forecast dispersion for inflation and output in two subperiods. Inflation_Disp_Q1 refers to one-quarter-ahead inflation forecast dispersion, Inflation_Disp_6Y refers to six-years-ahead inflation forecast dispersion, similarly for other variables. One-quarter to five-quarters-ahead forecasts are monthly and six-years-ahead forecasts are semiannually. Survey data are from the BCFF, and dispersions are in annual percentages.

difference of these two.[10]

Given the fact that, starting from around 1999, inflation ambiguity has switched from upward sloping to downward sloping and inflation shocks have switched from bad news to good news for future growth, we assume that the model has an unexpected discrete regime shift at the end of 1999 (for a detailed discussion, see Section 4). This is also consistent with the literature for regime breaks; for example, Campbell et al. (2014) argue that the first subperiod is the inflation fighting period of Volcker and Greenspan and the second subperiod is the recent period of low inflation and increased central bank transparency. Therefore $\mu_\pi^a$ is positive for the first subperiod (dispersion is bigger for a longer horizon) and negative for the second subperiod (dispersion is smaller for a longer horizon). $\mu_c^a$ is negative for both subperiods.

One concern is that the specification in equation (3) implies an upward- or downward-sloping trend in ambiguity. However, there seems to be no obvious trends in the realized ambiguity for inflation and GDP growth - the one-quarter-ahead forecast dispersion in Figure 1 and Figure 2. To understand this difference, equation (3) can be decomposed into two parts. The first part is a random walk with no drift (or constant given that the calibrated shocks are very small), which represents agents' ambiguity about the observation equation in the state space model of equation (1). Denoting the first part by $a_{1c,t}$ or $a_{1\pi,t}$, the alternative one-step-ahead distribution for $\Delta g_{t+1}$ and $\pi_{t+1}$ in equation (2) is now generated by $\tilde{\mu}_{c,t} \in [\mu_c - a_{1c,t}, \mu_c + a_{1c,t}]$ and $\tilde{\mu}_{\pi,t} \in [\mu_\pi - a_{1\pi,t}, \mu_\pi + a_{1\pi,t}]$. Denoted by $a_{2c,t}$ or $a_{2\pi,t}$, the second part containing the trend component captures agents' ambiguity about the state variables $x_{c,t+1}$ and $x_{\pi,t+1}$. And the alternative one-step-ahead distributions for $x_{c,t+1}$ and $x_{\pi,t+1}$ are generated by two sets of conditional means: $[\rho_c x_{c,t} - a_{2c,t}, \rho_c x_{c,t} + a_{2c,t}]$ and $[\rho_\pi x_{\pi,t} - a_{2\pi,t}, \rho_\pi x_{\pi,t} + a_{2\pi,t}]$. At each time period $t$, $x_{c,t}$ and $x_{\pi,t}$ are realized under the reference measure. Thus the one-step-ahead ambiguity for inflation and GDP growth (or the realized ambiguity), which is measured by one-quarter-ahead forecast dispersion

[10] We can modify the process by allowing for output growth shocks and inflation shocks. However, due to the CRRA utility, we show in an earlier version that these shocks have very small effects on term premium and yields.

in the data, contains only the first part ($a_{1c,t}$ or $a_{1\pi,t}$) and no trends. Only when agents evaluate future prospects that are more than one step ahead, does the second part matter. For example, ambiguity about $\Delta g_{t+2}$ contains both ambiguity about $x_{c,t+1}(a_{2c,t})$ and ambiguity about the observation equation ($a_{1c,t+1}$).[11]

As shown in Section 3, for bonds with maturities longer than one quarter, the second part ambiguity is the most important driver of yields. Even though bond and stock prices are solved under the worst-case distribution where the expectations hypothesis roughly holds, the model is simulated using the reference distribution. This difference makes excess returns on long-term bonds predictable. Note that we focus on the average pattern of bond and equity yields in this paper. To infer the historical performance of the model, we can use historical one-quarter-ahead dispersion as a measure for the size of ambiguity (only the first part) in the model. This specification of ambiguity is consistent with a recent finding that the estimated ambiguity is very persistent; for example, Dew-Becker and Bidder (2016) estimate the ambiguity shocks have a half-life of 70 years.[12]

*2.4. Preference: Recursive multiple priors*

PS 2007 show the importance of the Epstein and Zin (1989) preference to generate an upward-sloping nominal yield curve. To illustrate the key role of ambiguity yields, we assume investors have recursive multiple priors preference axiomatized by Epstein and Schneider (2003), but with CRRA utility function (investors are indifferent between early or late resolution of uncertainty):

$$V_t(C_t) = \min_{p_t \in \mathcal{P}_t} \mathbb{E}_{p_t} \left( U\left(C_t\right) + \beta V_{t+1}(C_{t+1}) \right) \tag{4}$$

[11] Alternatively, we can think of the first part as ambiguity about the reference distribution in equation (1), and assume that agents don't have exact knowledge about the ambiguity process in the first part. Then the second part containing the trend component captures agents' uncertainty about the ambiguity process in the first part. At each point of time, equation (3) represents agents' beliefs of how the size of ambiguity evolves over time when they evaluate future prospects (the model is solved under this measure). However, the realized ambiguity is generated by the reference (or true) ambiguity process containing only the first part, and the trend part is not materialized.

[12] Results in the model rely mainly on the second part ambiguity. We can change the first part ambiguity to a stationary process and the main results still hold.

where $U(C_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma}$, $\gamma$ is the coefficient of risk aversion, and $\beta$ reflects the investor's time preference.

*The worst-case belief*

The agent evaluates his expected lifetime utility under the subjective belief $p_t \in \mathcal{P}_t$, and the set of one-step-ahead beliefs $\mathcal{P}_t$ consists of the measures $p_t^{\tilde{\mu}}$ generated in Section 2.2. Because investors are ambiguity averse, they act pessimistically and evaluate future prospects under the worst-case measure. We use output growth as the endowment, and the worst-case measure for output growth associated with the minimum utility is generated by the distribution with $-a_{c,t}$ (the worst mean at each period).[13] For the worst-case inflation measure, it depends on the correlation between inflation expectations and worst-case expected real output growth. Using the bottom 10 average of individual GDP growth forecasts from the BCFF survey as the worst-case expected real growth, we find it is negatively associated with inflation expectations in the first subperiod and positively associated with inflation expectations in the second subperiod. The pattern is the same for all different measures of inflation expectation from the BCFF survey: top 10 average, median, and bottom 10 average of the individual inflation forecasts (the correlations are $-0.61$, $-0.52$, and $-0.39$, respectively, for the first subperiod, and 0.18, 0.38, and 0.46, respectively, for the second subperiod). Thus the worst-case inflation measure is generated by distribution with the highest mean inflation $+a_{\pi,t}$ in the first subperiod and the lowest mean inflation $-a_{\pi,t}$ in the second subperiod. In equilibrium, the "min" operator in the preference can be replaced by the worst-case measure.

[13] See Epstein and Wang (1994) for a proof.

*2.5. Asset markets*

To solve the model, we first rewrite the economy dynamics in vector forms:

$$\begin{aligned} z_{t+1} &= \phi_a a_t + \mu_z + x_{z,t} + \sigma^z \varepsilon_{t+1} \\ x_{z,t+1} &= \rho_x x_{z,t} + \sigma^x \varepsilon_{t+1} \\ a_{t+1} &= \mu_a + a_t + \sigma^a \varepsilon^a_{t+1} \end{aligned} \tag{5}$$

where $z_t = (\Delta g_t, \pi_t)^T$, $x_t = (x_{c,t},\, x_{\pi,t})^T$, and $a_t = (a_{c,t},\, a_{\pi,t})^T$. All other parameters are in vector forms that are consistent with the earlier specification in Section 2. Note that equation (5) describes the worst-case measure in equilibrium. $\phi_a$ represents the equilibrium choice of the upper or lower bound, equal to $-1$ or $+1$. In the following two subsections, we will solve bond yields and equity yields using vector forms.

*2.5.1. Bond price*

Since the representative agent forms expectations under the worst-case measure when making portfolio choices, the Euler equation holds under the worst-case measure. Given the CRRA utility function, the log nominal pricing kernel or the nominal stochastic discount factor can be written as

$$m^{\$}_{t,t+1} = log\beta - \gamma \Delta g_{t+1} - \pi_{c,t+1} = log\beta - v' z_{t+1} \tag{6}$$

where $v' = (\gamma,\, 1)$. The time-$t$ price of a zero-coupon bond that pays one unit of consumption $n$ periods from now is denoted $P^{(n)}_t$, and it satisfies the recursion

$$P^{(n)}_t = E_{p^o_t}[M^{\$}_{t,t+1} P^{(n-1)}_{t+1}] \tag{7}$$

with the initial condition that $P^{(0)}_t = 1$ and $E_{p^o_t}$ is the expectation operator for the worst-case measure. Given the linear Gaussian framework, we assume that $p^{(n)}_t = log(P^{(n)}_t)$ is a linear function of $a_t$ and $x_t$:

$$p_t^{(n)} = -A^{(n)} - B^{(n)} x_t - C^{(n)} a_t. \tag{8}$$

When we substitute $p_t^{(n)}$ and $p_{t+1}^{(n-1)}$ in the Euler equation (7), the solution coefficients in the pricing equation can be solved with $B^{(n)} = B^{(n-1)}\rho_x + v' = v' \left(\sum_{i=o}^{n-1}(\rho_x)^i\right)$, $C^{(n)} = C^{(n-1)} + v'\phi_a = v'\phi_a n$, and $A^{(n)}$ is given in the appendix. The log holding period return from buying an $n$ period bond at time $t$ and selling it as an $n-1$ period bond at time $t+1$ is defined as $r_{n,t+1} = p_{t+1}^{(n-1)} - p_t^{(n)}$, and the subjective excess return is $er_{n,t+1} = -Cov_t\left(r_{n,t+1}, m_{t,t+1}^{\$}\right) = -B^{(n-1)}\sigma^x \sigma^{z'} v$.

As we can see from the solution, the yield parameter for ambiguity is constant over horizons $n$, and the average $x_{z,t}$ is zero, implying that, on average, expected growth and inflation do not affect long-term bond yields. The channel through which ambiguity affects bond yields is the expected future interest rate embedded in $A^{(n)}$ (due to the trend component $\mu_a$, $A^{(n)}/n$ is bigger for a longer horizon). To solve the price and yields for real bonds, we can just replace $v'$ with $v' = (\gamma, 0)$.

*2.5.2. Stock price*

Equity price and returns can be solved using the real stochastic discount factor $m_{t,t+1} = log\beta - \gamma\Delta g_{t+1}$. For any asset $j$ with a real payoff, the first-order condition yields the following asset pricing Euler condition:

$$E_{p_t^o}[exp(m_{t,t+1} + r_{j,t+1})] = 1 \tag{9}$$

where $E_{p_t^o}$ is the expectation operator for the worst-case measure, and $r_{j,t+1}$ is the log of the gross return on asset $j$.

To solve the market return, it is assumed that the log price-dividend ratio for dividend claims, $z_t$, is linear in $a_{c,t}$ and $x_{c,t}$:

$$z_t = A_0 + A_1 x_{c,t} + A_2 a_{c,t}. \tag{10}$$

The log market return is given by the Campbell and Shiller (1988) approximation

$$r_{m,t+1} = k_0 + k_1 z_{t+1} + \Delta d_{t+1} - z_t \tag{11}$$

where $k_0$ and $k_1$ are log linearization constants, which will be discussed with more detail in the appendix. By substituting (10) and (11) into the Euler equation (9), we can solve $A_0$, $A_1$, and $A_2$ with $A_1 = \frac{\zeta_d - \gamma}{1-k_1\rho_c}$ and $A_2 = -\frac{\zeta_d-\gamma}{1-k_1}$.

For the price of individual dividends (or dividend strips), we can solve it in a similar way. Let $P_{t,n}$ denote the price of a dividend at time $t$ that is paid $n$ periods in the future. Let $D_{t+1}$ denote the realized dividend in period $t+1$. The price of the first dividend strip is given by $P_{t,1} = E_{p_t^o}[M_{t,t+1}D_{t+1}] = D_t E_{p_t^o}[M_{t,t+1}\frac{D_{t+1}}{D_t}]$, and the recursion $P_{t,n} = E_{p_t^o}[M_{t,t+1}P_{t+1,n-1}]$ allows us to compute the remaining dividend strip prices. Given the linear Gaussian framework, we assume that the log dividend strip prices, scaled by the current dividend, are also affine in the state variables:

$$pd_t^{(n)} = A_0^{(n)} + A_1^n x_{c,t} + A_2^{(n)} a_{c,t} \tag{12}$$

Similar to the bond prices, we can first compute $pd_t^{(1)}$ using $pd_t^{(1)} = log\left(E_{p_t^o}[M_{t,t+1}\frac{D_{t+1}}{D_t}]\right)$, and then use the recursion $pd_t^{(n)} = log\left(E_{p_t^o}[exp\left(m_{t,t+1} + \Delta d_{t+1} + pd_{t+1}^{(n-1)}\right)]\right)$ to compute the remaining dividend strip prices. The solution coefficients in the pricing equation (12) are $A_1^n = A_1^{n-1}\rho_c + \zeta_d - \gamma = (\zeta - \gamma)\left(\sum_{i=o}^{n-1}(\rho_c)^i\right)$, $A_2^{(n)} = A_2^{(n-1)} - (\zeta_d - \gamma) = -n\,(\zeta_d - \gamma)$, and $A_0^{(n)}$ is given in the appendix. Dividend yield or equity yield is defined as $ey_t^n = -\frac{1}{n}pd_t^{(n)}$, which is downward sloping as $-\frac{1}{n}A_0^{(n)}$ is downward sloping (due to the trend component $\mu_c^a$). The logic is the same for bond yields where average $x_{z,t}$ is zero and $-\frac{1}{n}A_2^{(n)}$ is constant.

It is worth mentioning that although the ambiguity averse agent acts pessimistically and prices assets under the worst-case measure, we are interested in expected returns under the reference model because it is the best estimate of the data generating process based on historical data, which are the counterpart of the observed expected returns. The wedge between reference and worst-case mean growth makes the model-implied expected return bigger (ambiguity premium). Solutions are provided in the appendix.

## 3. Empirical findings

Given the analytical solutions, in this section we can calculate the nominal/real bond yields, dividend yields, and volatility explicitly. To be consistent with our empirical finding that the slope of the yield curve for inflation ambiguity has switched from positive to negative, the whole sample, 1985.Q1 to 2017.Q4, is broken into two subperiods consistent with major shifts in monetary policy. Because the earliest available data for the BCFF forecast dispersion is 1985.Q1, our first subperiod covers 1985.Q1 to 1999.Q4, part of the Fed chairmanships of Paul Volcker and Alan Greenspan. The second subperiod, 2000.Q1 to 2017.Q4, covers the later part of Greenspan's chairmanship and the earlier part of Bernanke's chairmanship. We assume that transitions from one regime to another are structural breaks, completely unanticipated by investors. In Section 4, we discuss the model implications of allowing a more gradual transition between these two regimes.

### *3.1. Data*

We use quarterly US data on output growth, inflation, interest rates, and forecast dispersion from 1985.Q1 to 2017.Q4. Real output growth and CPI inflation are from the Bureau of Economic Analysis. The forecast dispersion for real output growth and CPI inflation are from the Blue Chip Financial Forecast survey. The end-of-quarter yields for one- to ten-year bonds are from the daily dataset constructed by Gürkaynak et al. (2007) (GSW 2007). The TIPS yields and end-of-quarter yields for three-month Treasury bills are from the U.S. Department of the Treasury via the Fed database at the St. Louis Federal Reserve, which are available from 2003 to 2017. For the one-quarter real risk-free rate, we follow Beeler and Campbell (2012) and create a proxy for the ex-ante risk-free rate by forecasting the ex-post quarterly real return on three-month Treasury bills with past one-year inflation and the most recent available three-month nominal bill yield.

*3.2. Estimation and calibration*

The state space system for output growth and inflation is estimated using maximum likelihood separately for each subperiod. The resulting parameter values are reported in Table 2. The correlation between output growth and inflation is captured by $\sigma^x_{c\pi}$, which is negative for the first subperiod and positive for the second subperiod. Consistent with PS 2007, inflation shocks were bad news for future growth in the first subperiod, however, they turned to be good news in the second subperiod. At the same time, worst-case expected real growth is negatively associated with inflation expectation in the first subperiod and positively associated with inflation expectation in the second subperiod. Thus, for ambiguity averse investors, the worst-case inflation measure is the upper bound in the first subperiod and is the lower bound in the second subperiod.

The volatility parameters in the ambiguity process are calibrated to match their counterparts in dispersion data. For example, within each subperiod, $\sigma^{a\pi}_a$ is chosen to match one-quarter-ahead inflation forecast dispersion volatility, $\sigma_{ac}$ is chosen to match one-quarter-ahead output growth forecast dispersion volatility, and $\sigma^{ac}_a$ is chosen to match the correlation between one-quarter-ahead worst-case inflation and one-quarter-ahead worst-case output growth. Table 2 shows that these values are quantitatively small, and we actually show in the following section that the impact of volatility in the ambiguity process on bond yields is negligible in this model. Given the small volatility, our results are quantitatively close to the extreme case where there is no uncertainty in the ambiguity process.

The trend component $\mu_a$ and the initial value $a_0$ are also calibrated to match the data in dispersion. For each subperiod, $a_{c,0}$ and $a_{\pi,0}$ are chosen to match average one-quarter-ahead dispersion values in the data, $\mu^a_c$ and $\mu^a_\pi$ are chosen to match the average difference between six-years-ahead and one-quarter-ahead forecast dispersion (six-years-ahead minus one-quarter-ahead dispersion and then divide by 24).

For other parameters, we follow the literature and set risk aversion as 3, and set leverage parameter $\zeta_d = 3$. $\mu_d$ is chosen such that the average rate of dividend growth is equal to the mean growth rate of dividends in the data. Given the leverage ratio, $\sigma_d$ can

| State Space Model | $\mu_c$ | $\mu_\pi$ | $\rho_c$ | $\rho_\pi$ | $\sigma_c$ | $\sigma_\pi$ | $\sigma_c^x$ | $\sigma_\pi^x$ | $\sigma_{c\pi}^x$ |
|---|---|---|---|---|---|---|---|---|---|
| 85.Q1–99.Q4 | 0.86 | 0.74 | 0.92 | 0.76 | 0.46 | 0.27 | -0.03 | 0.10 | -0.07 |
| 00.Q1–17.Q4 | 0.45 | 0.57 | 0.49 | -0.19 | 0.55 | 0.42 | 0.21 | 0.17 | 0.03 |
| Ambiguity | $\mu_c^a$ | $\mu_\pi^a$ | $\sigma_{ac}$ | $\sigma_a^{ac}$ | $\sigma_a^{a\pi}$ | $a_{0,c}$ | $a_{0,\pi}$ | | $\beta$ |
| 85.Q1–99.Q4 | -0.0044 | 0.0029 | 0.012 | -0.009 | 0.010 | 0.28 | 0.19 | | 1.0115 |
| 00.Q1–17.Q4 | -0.0043 | -0.0063 | 0.009 | 0.0047 | 0.015 | 0.22 | 0.26 | | 1.0054 |
| Other | $\gamma$ | $\zeta_d$ | $\mu_d$ (P1) | $\mu_d$ (P2) | $\sigma_d$ (P1) | $\sigma_d$ (P2) | | | |
| | 3 | 3 | -2.40 | -1.17 | 2.70 | 2.55 | | | |

Table 2: Configuration of model parameters

Table 2 reports output growth, dividend growth, inflation, and ambiguity processes parameters. All parameters are given in quarterly terms. Mean and standard deviation are in percentages.

be calibrated to match the standard deviation of dividend growth in the data. Finally, time preference $\beta$ is calibrated to match one-year nominal yields in the data for each subperiod, which are close to the value in PS 2007.[14]

### *3.3. Bond yields and volatility*

#### *3.3.1. Real bond yields*

Using TIPS data from the U.S. Department of the Treasury from 2003 to 2017, Table 3 reports the level and volatility of real yields. Although there are less than twenty years of TIPS data, the observed slope has never been quantitatively significantly negative. The volatility of real yields is smaller for a longer horizon. Campbell (1986) argues that, if consumption growth is modeled as a persistent process where positive shocks cause upward revisions in expected future growth, a positive consumption shock causes real interest rates to increase and bond prices to fall. In this case, real bonds hedge consumption risk and have a negative real term premium. Thus, asset pricing models with persistent consumption growth processes are likely to be inconsistent with the data.

In this model, for both subperiods, investors are less ambiguous about longer horizon output growth. In equilibrium, ambiguity averse agents choose the lower bound from

[14] Higher time preference helps to lower bond yield levels. We can also set $\beta$ to be smaller than 1, but then we need to either decrease the risk aversion parameter or change the level of ambiguity to match the bond yield level.

| Real Bond | 00.Q1–16.Q4 | 1Q | 5Y | 7Y | 10Y |
|---|---|---|---|---|---|
| **Data** | Yield | | 0.57 | 0.86 | 1.10 |
| | Std | | 1.02 | 0.95 | 0.90 |
| **Model** | Yield | 0.55 | 0.94 | 1.14 | 1.44 |
| | Std | 1.93 | 0.51 | 0.50 | 0.49 |
| **Model (No ambiguity)** | Yield | 3.19 | 3.09 | 3.08 | 3.08 |
| | Std | 1.87 | 0.18 | 0.13 | 0.09 |

Table 3: Real bond yields and volatility
This table presents data and model-implied real bond yields and volatility for the second subperiod. TIPS yields are available for five years, seven years, and ten years to maturity from 2003 and 2017.

the set of alternative mean output growth rates, which are upward sloping. As a result, the future interest rates are higher for longer horizons. The model-implied real yields are reported in Table 3, which are upward sloping and consistent with the data. The volatility in yields consists of two parts: (1) shocks from expected growth where the weight is smaller for a longer horizon (due to the persistence in expected output growth $\rho_c$), and (2) shocks from ambiguity where the weight is constant. Therefore our model-implied volatility is consistent with the data and is downward sloping. However, due to our small risk aversion parameter, the size of volatility is somewhat smaller in magnitude.

To check the effectiveness of the mechanism described above, we shut down the ambiguity for output growth and report the results for real yield in Table 3 as well. As expected, the real yield curve is almost flat now (the higher yield for one-quarter real bonds is due to the fact that the real interest rate provides hedges to growth risks. But with CRRA utility, this effect only appears in short horizons), and the volatility also rapidly declines to almost zero (due to a small $\rho_c$ and no ambiguity shocks in the long end of the yield curve).

### *3.3.2. Nominal bond yields*

There is a large body of finance literature modeling bond yields without distinguishing differences between subperiods. Most of these studies use inflation non-neutrality established in PS 2007 to generate upward-sloping yield curves. This mechanism requires that agents prefer early resolution of uncertainty, and at the same time, inflation is bad news for future growth. A positive surprise to inflation implies lower future growth and lower real payoff of long-term bonds. Therefore, agents require excess returns to hold long-term bonds over short-term bonds. To understand the changes in correlation between U.S. Treasury bond returns and stock returns, recent studies have shown that the correlation between consumption growth and inflation has switched from negative to positive after the late 1990s. For example, Song (2017) estimates a regime switch version of PS 2007 and finds that the U.S. economy entered a positive correlation regime (between inflation and growth) after the late 1990s and largely remained in that regime throughout the sample. Our estimation for the reference state space model in Table 2 is also consistent with these findings. Given these changes, the standard approach implies a downward-sloping nominal yield curve for the current period. However, we still observe an upward-sloping nominal yield curve in the data (as reported in Table 4), which implies that we need to understand nominal yields using a different approach, at least for the current period.

During the first subperiod in this model, investors have more ambiguity about inflation in longer horizons. Together with the fact that the worst-case expected growth is negatively associated with inflation expectation, ambiguity averse investors choose the upper inflation bound to evaluate the future perspective. This implies that expected inflation in equilibrium is upward sloping, which generates an upward-sloping nominal yield curve. During the second subperiod, the worst-case expected growth is positively associated with inflation expectation, and the worst-case mean inflation becomes the lower bound. At the same time, investors have less ambiguity about inflation in longer horizons, which again implies an upward-sloping mean inflation in equilibrium. Therefore the model generates upward-sloping nominal yield curves in both subperiods, but with a different mechanism. Table 4 reports nominal bond yields from the data and implied by the model for both subperiods, and it is clear that the model matches the data very well.

Another important difference in nominal yields is that the average yield level has dropped dramatically from 6.14 for a one-year nominal bond in the first subperiod to 1.86 in the second subperiod. Part of the reason for this change is the decrease in mean output growth (from 0.86% quarterly to 0.45% quarterly) and decrease in mean inflation (from 0.74% quarterly to 0.57% quarterly). They alone (including differences in time preference for the two subperiods), however, are far from providing a complete answer to the almost 70% drop in nominal yields. In this model, the worst-case mean inflation in equilibrium is the upper bound in the first subperiod and switches to the lower bound in the second subperiod. Thus, the difference between the upper bound and the lower bound of the inflation dispersion provides another significant contribution to the drop in nominal yields (accounting for 42% of the changes).

In a similar way to the real bonds, nominal bond yield volatility consists of both volatility from the expected growth $x_{z,t+1}$, which is decreasing over horizons, and volatility from the ambiguity process $a_{t+1}$, which is constant over horizons. Thus the model-implied volatility shares the same pattern of decreasing over horizons as in the data. However, the size of volatility is somewhat smaller in magnitude. Besides the small risk aversion parameter as one reason, we can also increase the ambiguity volatility in order to increase the bond yield volatility.[15]

*Without ambiguity*

Since the nominal interest rate is the sum of the real interest rate and expected inflation, and given the upward-sloping real yield curve (due to ambiguity about output growth), it is natural to ask whether inflation ambiguity matters for generating upward-sloping nominal yield curves. For this purpose, we shut down the ambiguity for inflation only and provide the yields and volatility in Table 4. There are two main differences: (1) the slopes (ten-year yield - one-year yield) for both subperiods are smaller without inflation ambiguity (0.93 vs. 1.13 for Period 1 and 0.87 vs. 1.32 for Period 2), which is due to the trend in inflation ambiguity; and (2) the yield level is smaller/bigger for the first/second subperiod because of investors' different worst-case inflation choices.

To check the overall effectiveness of ambiguity for both inflation and output growth, we

[15] Currently we match the ambiguity volatility in the model to volatility in the dispersion data. We can also calibrate ambiguity volatility using yield volatility.

| | Nominal Bond | 1Y | 2Y | 3Y | 4Y | 5Y | 10Y |
|---|---|---|---|---|---|---|---|
| Period 1 | 1985–1999 | | | | | | |
| **Data** | Yield | 6.14 | 6.47 | 6.69 | 6.87 | 7.01 | 7.47 |
| | Std | 1.56 | 1.51 | 1.48 | 1.46 | 1.45 | 1.39 |
| **Model** | Yield | 6.17 | 6.30 | 6.43 | 6.56 | 6.68 | 7.30 |
| | Std | 1.15 | 1.09 | 1.03 | 0.99 | 0.95 | 0.87 |
| **Model (No inflation ambiguity)** | Yield | 5.39 | 5.50 | 5.61 | 5.71 | 5.82 | 6.32 |
| | Std | 1.08 | 1.00 | 0.94 | 0.90 | 0.86 | 0.77 |
| **Model (No ambiguity)** | Yield | 8.67 | 8.68 | 8.68 | 8.68 | 8.68 | 8.67 |
| | Std | 0.81 | 0.71 | 0.62 | 0.54 | 0.48 | 0.29 |
| Period 2 | 2000–2017 | | | | | | |
| **Data** | Yield | 1.83 | 2.06 | 2.31 | 2.57 | 2.81 | 3.68 |
| | Std | 1.86 | 1.76 | 1.64 | 1.55 | 1.47 | 1.28 |
| **Model** | Yield | 1.84 | 1.97 | 2.11 | 2.26 | 2.41 | 3.16 |
| | Std | 1.16 | 0.81 | 0.72 | 0.68 | 0.66 | 0.64 |
| **Model (No inflation ambiguity)** | Yield | 2.84 | 2.92 | 3.01 | 3.11 | 3.21 | 3.71 |
| | Std | 1.09 | 0.71 | 0.59 | 0.55 | 0.53 | 0.49 |
| **Model (No ambiguity)** | Yield | 5.41 | 5.38 | 5.37 | 5.36 | 5.36 | 5.35 |
| | Std | 0.98 | 0.52 | 0.34 | 0.26 | 0.21 | 0.10 |

Table 4: Nominal bond yields and volatility

This table presents data and model-implied nominal bond yields and volatility for both subperiods. The end-of-quarter yields for one- to ten-year bonds are from the daily dataset constructed by GSW 2007.

shut down the ambiguity for both inflation and output growth. The results are provided in Table 4. As expected, the same as for real bonds, the nominal yield curve is almost flat for both subperiods, and volatility also rapidly declines to almost zero. The yield levels are also higher for both subperiods under the reference measure with higher mean output growth (output growth ambiguity dominates inflation ambiguity in determining yield level because of the risk aversion parameter $\gamma > 1$).

*3.4. Expectations hypothesis and predictability of bond returns*

The expectations hypothesis states that the yield for an $n$ periods bond is the average of expected future one-period bond yields. Let $y_t^{(n)} = -\frac{1}{n}p_t^{(n)}$ denote the yield for an $n$ periods bond at time $t$. The intuition of the expectations hypothesis can be illustrated by the following two-periods example where $2y_t^2 = y_t^1 + E_t\left(y_{t+1}^1\right)$. If the yield curve is upward sloping as in the data, $y_t^1 < y_t^2$, it must be that $y_t^1 < y_t^2 < E_t\left(y_{t+1}^1\right)$, that is, the short rate will rise. However, the realized future short rate does not increase enough in the data, and the expectations hypothesis does not seem to work well. The expectations hypothesis is often formally tested through the following equation:

$$y_{t+1}^{n-1} - y_t^n = \alpha + \beta_n\left(\frac{y_t^n - y_t^1}{n-1}\right) + \epsilon_{t+1}. \tag{13}$$

The expectations hypothesis implies that $\beta_n = 1$. However, in the data, many studies (for example, Campbell and Shiller (1991)) show that $\beta_n < 1$, is often negative, and is decreasing with the horizon $n$.

Using survey expectations instead of realized future yields as subjective beliefs, other literature (Froot (1989); Piazzesi et al. (2015)) shows that the failure of the expectations hypothesis is due to expectational errors. For the two-periods example, the survey expectations of the short rate tomorrow $E_t^s\left(y_{t+1}^1\right)$ is close to $E_t\left(y_{t+1}^1\right)$ and higher than the subsequently realized short rate; thus the expectations hypothesis is violated using the realized yields.[16] In this model, agents evaluate future prospects under the worst-

[16]Cieslak (2018) shows the federal fund rate expectations measured by the BCFF survey mean are on

case belief, which implies that future expected inflation and real growth are higher and higher, hence future expected short nominal and real rates are higher and higher. Given CRRA utility, the subjective bond premium $er_{n,t+1}$ is close to zero, and the expectations hypothesis roughly holds.[17] Nevertheless, at each time period $t$, $x_{c,t}$ and $x_{\pi,t}$ are realized under the reference measure, and the realized ambiguity contains only the random walk part ($a_{1c,t}$ or $a_{1\pi,t}$) with no trend (the trend part is not materialized). Hence, consistent with the empirical evidence, the realized short rates are lower than expected as in the worst-case belief, which also makes excess returns on long-term bonds predictable.

To formally assess the expectations hypothesis, we show in the appendix that the difference between the left-hand side and right-hand side of equation (13) is $\left(y_{t+1}^{n-1} - y_t^n\right) - \frac{y_t^n - y_t^1}{n-1} = v'\phi_a\left((a_{t+1} - a_t) - \mu_a\right) + \frac{VarCov_{n-1}}{n-1}$.[18] Given that $\frac{VarCov_{n-1}}{n-1}$ is quantitatively very small, the difference is mainly driven by $v'\phi_a\left((a_{t+1} - a_t) - \mu_a\right)$. From equation (5) we know the worst-case belief $a_{t+1} = \mu_a + a_t + \sigma^a\varepsilon_{t+1}^a$, which makes the difference $v'\phi_a\left((a_{t+1} - a_t) - \mu_a\right) = v'\phi_a\sigma^a\varepsilon_{t+1}^a$. The shock $v'\phi_a\sigma^a\varepsilon_{t+1}^a$ can then be moved into the error term $\epsilon_{t+1}$, thus $\beta_n \approx 1$ for all maturities under the worst-case belief, and the expectations hypothesis roughly holds.

However, as discussed in Section 2.3, because ambiguity about $x_{c,t}$ and $x_{\pi,t}$ do not materialize at time $t$, the realized ambiguity contains only the random walk part with no trend, $a_{t+1} = a_t + \sigma^a\varepsilon_{t+1}^a$. In this case, $v'\phi_a\left((a_{t+1} - a_t) - \mu_a\right) = -v'\phi_a\mu_a < 0$ ignoring the Gaussian shock. Taking advantage of the closed form solution, we show in the appendix that the coefficient $\beta_n$ in equation (13) would be $-1$ for all $n$ if we ignore $x_{z,t}$ and a variance/covariance term. Because of the low autocorrelation ($\rho_x$), short-term yields are more sensitive to $x_{z,t}$ and the variance/covariance term, yet $\beta_n$ for long maturities are mainly driven by the difference above and are close to $-1$. This intuition can be confirmed by the regression results reported in Table 5. As in the data, the slope coefficients of the

average higher than the subsequently realized federal fund rate.

[17] The subjective bond premium $er_{n,t+1}$ is less than 0.1% in absolute values for all maturities and both subperiods.

[18] Note that all Gaussian shocks can be thought of as the error term in equation (13) and hence are not included in this difference.

| Data | Two years | Three years | Four years | Five years |
|---|---|---|---|---|
| EH slope | -0.41 | -0.78 | -1.14 | $-1.15$ |
| CP slope | 0.44 | 0.85 | 1.28 | 1.43 |
| CP $R^2$ | 0.15 | 0.17 | 0.20 | 0.17 |
| | | | | |
| Model | Two years | Three years | Four years | Five years |
| EH slope | -0.43 | -0.61 | -0.79 | -0.98 |
| CP slope | 0.79 | 0.93 | 1.07 | 1.21 |
| CP $R^2$ | 0.24 | 0.17 | 0.14 | 0.11 |

Table 5: Predictability of bond returns

This table presents the slopes in the expectations hypothesis (EH) regressions, and the slopes and $R^2$s in CP 2005 single-factor bond premium regressions for the whole sample. The model-implied statistics displayed are the median values from ten thousand finite sample simulations of equivalent length to the dataset (from 1985.Q1 to 2017.Q4). The end-of-quarter one- to five-year bond yields are from the Center for Research in Security Prices' monthly Treasury Fama-Bliss discount bond yields.

model simulation in the expectations hypothesis projections are negative and decreasing with maturity.[19] The coefficients for long maturity bonds become closer to $-1$ as the effects of $x_{z,t}$ and the variance/covariance term vanish.

To further evaluate the predictability of bond returns, we follow the approach in Cochrane and Piazzesi (2005) by first regressing the average of one-year nominal excess bond returns of two to five years to maturity on one- to five-year forward rates, extracting a single bond factor $\hat{rx}_t$ from this regression, and then forecasting excess bond returns at each maturity $n$ from two to five years, $rx^{n,\$}_{t\to t+1} = const + b_n \hat{rx}_t + error$. They show that the estimate $b_n$ is positive and increasing with horizons. Table 5 shows the slopes and $R^2$s of the regression using quarterly observations of US bond yields from 1969 to 2010 from Bansal and Shaliastovich (2013). This model shares a similar pattern and magnitude for the slopes as in the data. $R^2$s are close to the data in magnitude, but decrease with horizons in the model. In sum, the model matches well the bond return predictability evidence from both the expectations hypothesis regression and the single-factor regression of Cochrane and Piazzesi (2005).

[19]The expectations hypothesis (EH) slopes, Cochrane and Piazzesi (CP, 2005) slopes, and CP $R^2$ in the data are from Bansal and Shaliastovich (2013). They use quarterly observations of US bond yields from 1969 to 2010. Their sample period is almost identical to ours.

Overall, the expectations hypothesis roughly holds when expected future short rates are formed under the worst-case belief (equilibrium measure). Yet, the realized short rates are lower than expected as in the subjective equilibrium measure. In this model, the so-called term premium is mainly due to the above difference in future short rates, which is consistent with Piazzesi et al. (2015). Excess returns are predictable by current yields and forward rates because both of them are driven by the trend component in ambiguity.

*3.5. Historical yield, slope, and recession*

In this paper, we focus on the average nominal and real yield curves over the two subperiods, not their historical movements. Nevertheless, the historical slope of the Treasury yield curve has often been cited as a leading economic indicator with inversion of the curve being thought of as a signal of a recession (for example, Estrella and Hardouvelis (1991); Estrella and Mishkin (1996, 1998)). To shed light on the historical performance of the model, we can ask what the model would imply if we replaced the state variables in the pricing equations with real time data.

Both bond price and stock price can be expressed as a linear function of $a_t$ and $x_t$, with closed form solutions given in Section 2.5. In the data, the realized ambiguity is measured by one-quarter-ahead forecast dispersion for GDP growth and inflation. The expected GDP growth and inflation $x_t$ can be measured by median survey from the BCFF. To avoid potential noise, we use a 12-month rolling average of the survey median and dispersion. The data are then standardized such that the first two moments are consistent with the model.

Using the bond yield parameter values from the model, the upper panel in Figure 3 shows the comovements of the model-implied one-year nominal yields and the historical realized one-year Treasury yields, and the lower panel in Figure 3 shows the comovements of the model-implied ten-year minus one-year nominal spread and the historical realized slopes. The model-implied yields and slopes are significantly correlated with yields and slopes in the data where the correlations are 0.59 and 0.45 respectively. It is worth mentioning that the model-implied one-year nominal yields track the data very well except during these post financial crisis periods, where one-year Treasury yields are constrained by the zero lower bound.

*3.6. Equity yields*

Recent empirical findings regarding the price and return for individual dividends (or dividend strips) pose some challenges to current equilibrium asset pricing models (see Van Binsbergen and Koijen (2017) for a summary).[20] For example, using dividend future contracts for the S&P500 from 2002 to 2014, Van Binsbergen and Koijen (2017) show that the dividend future returns are slightly upward sloping and the volatility of forward equity yields is downward sloping. And the market returns are not significantly different from individual dividend spot returns. They argue that leading asset pricing models are not able to match those features in the data. Note that their empirical findings on equity yields are different for different countries. Since our model is estimated and calibrated using US data, we will focus on the findings from the S&P500.

Table 5 reports the model's market return, dividend spot return, dividend future return, and equity yield volatility. The closed form solution for the market return is given in Section 2, and the dividend spot return is defined as $log\left(P_{t+1,n-1}\right) - log\left(P_{t,n}\right) = pd_{t+1}^{(n-1)} - pd_t^{(n)} + \Delta d_{t+1}$. The dividend future return is the dividend spot return less the same horizon bond holding period return. Since the agent faces the same size of one-step-ahead ambiguity, the market return and dividend spot return are very close in this model, however, dividend yields are downward sloping because long horizon dividends feature less ambiguity. Because the holding period return for real bonds is downward sloping in our model, the dividend future returns is slightly upward sloping, which is consistent with Van Binsbergen and Koijen (2017). For the same reason as for the bond yields volatility, the forward equity yield volatility is downward sloping, which is consistent with the data.

[20] Also, Van Binsbergen et al. (2012) provide the first direct measurement of dividend strip prices using options data. Van Binsbergen et al. (2013) extend this evidence using dividend futures contracts.

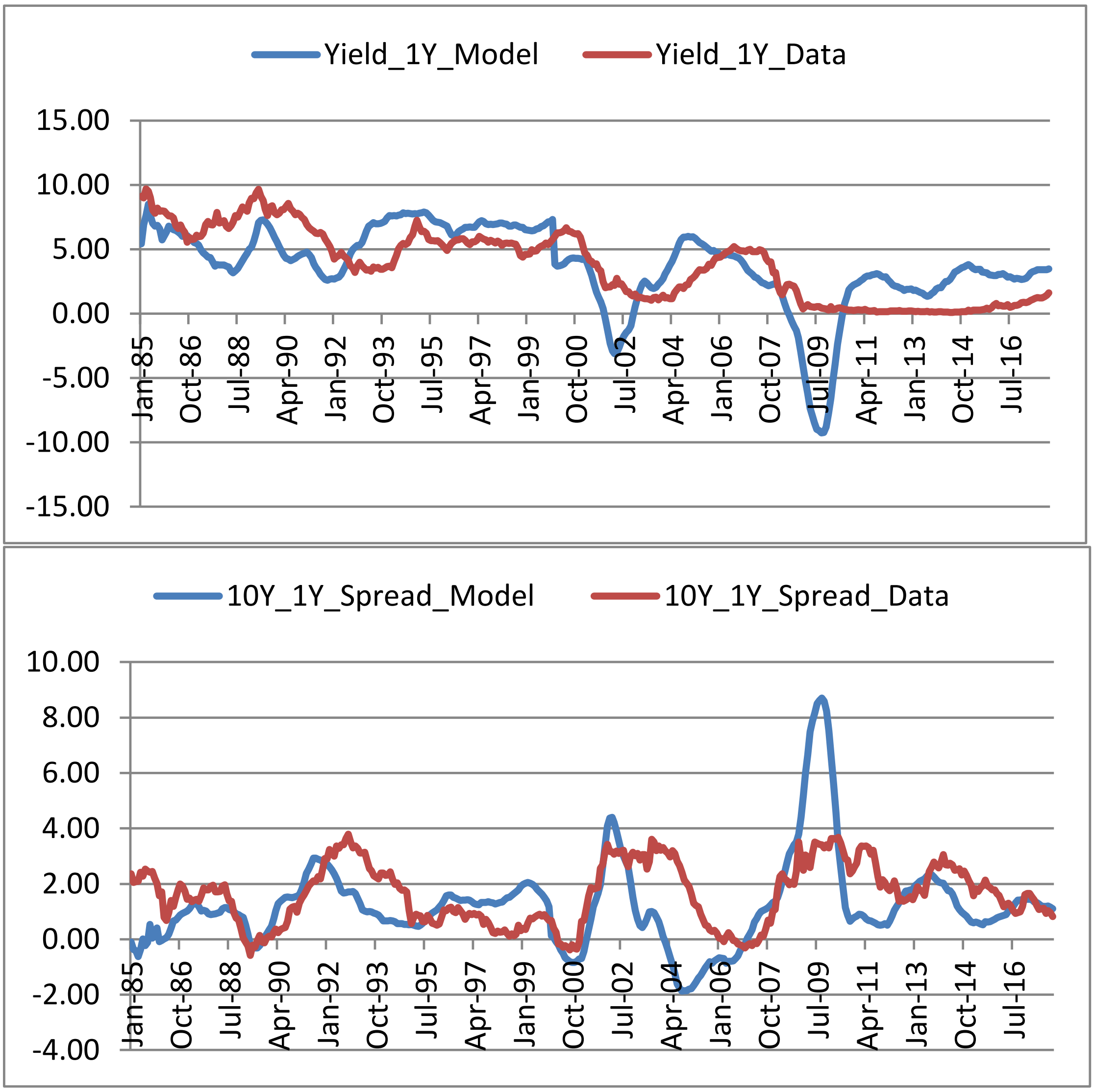


Figure 3: Yield, slope, and recession

The slope is the ten-year nominal yield minus the one-year treasury rate. The dispersion is calculated as a combination of slopes of term structure of inflation and real GDP forecast dispersion. All data are monthly from 1985 to 2017.

| **Model (Period 2)** | 1Q | 1Y | 5Y | 7Y | 10Y |
|---|---|---|---|---|---|
| Market Return | | | 7.73 | | |
| Dividend Spot Return | 7.73 | 7.73 | 7.73 | 7.73 | 7.73 |
| Dividend Future Return | 3.07 | 3.13 | 3.14 | 3.15 | 3.16 |
| Equity Yield | 7.03 | 7.02 | 6.95 | 6.91 | 6.86 |
| Equity Yield Volatility | 4.04 | 1.94 | 0.69 | 0.63 | 0.60 |

Table 6: Dividend strip return and volatility

This table presents the model-implied market return, dividend spot returns, dividend future returns, and forward equity yield volatility for the second subperiod. To calculate returns and volatility for dividend strips, as well as for market return, we set time preference $\beta = 0.995$ in order to have a stable approximation for the Campbell and Shiller approximation.

## 4. Robustness

This section provides further checks for the sensitivity of the results in several dimensions.

### *4.1. Regime shift and learning*

In this paper, we assume that there is an unexpected discrete regime shift at the end of 1999 for the following reasons: (1) the term structure of inflation forecast dispersion has switched from upward sloping to downward sloping after the late 1990s (see Figure 1); (2) Figure 4 shows that the correlation between worst-case GDP growth and inflation forecasts has switched from negative to positive after the late 1990s and has largely remained in that regime thereafter (this is also true for the top 10 average and bottom 10 average of individual inflation forecasts); and (3) this is consistent with the literature for regime breaks; for example, Campbell et al. (2014) argue that the first subperiod is the inflation fighting period of Volcker and Greenspan and the second subperiod is the recent period of low inflation and increased central bank transparency.

While it is useful to clarify the mechanics by assuming an unanticipated regime switch in the late 1990s, there seems no obvious event in this period that this could be tied to. We may ask what the model would imply if we allowed a more gradual transition between these two regimes. Suppose investors know the probabilities of each regime at time $t$; then stock and bond prices can be computed as the weighted average of the two solutions in Section 2. Given the fact that the probability of regime one (negative correlation between growth and inflation expectation) is very high before the late 1990s and close to zero thereafter (see, for example, the estimation in Song (2017)), the mechanism of this paper still works and the model results are quantitatively similar. Because the theoretical framework of learning under ambiguity with a regime switch is not clear yet, we will leave this case for future research.

### *4.2. Magnitude of ambiguity*

Given the specification for the ambiguity process, one natural question is whether the size of the ambiguity is reasonable. We use the error detection probability approach

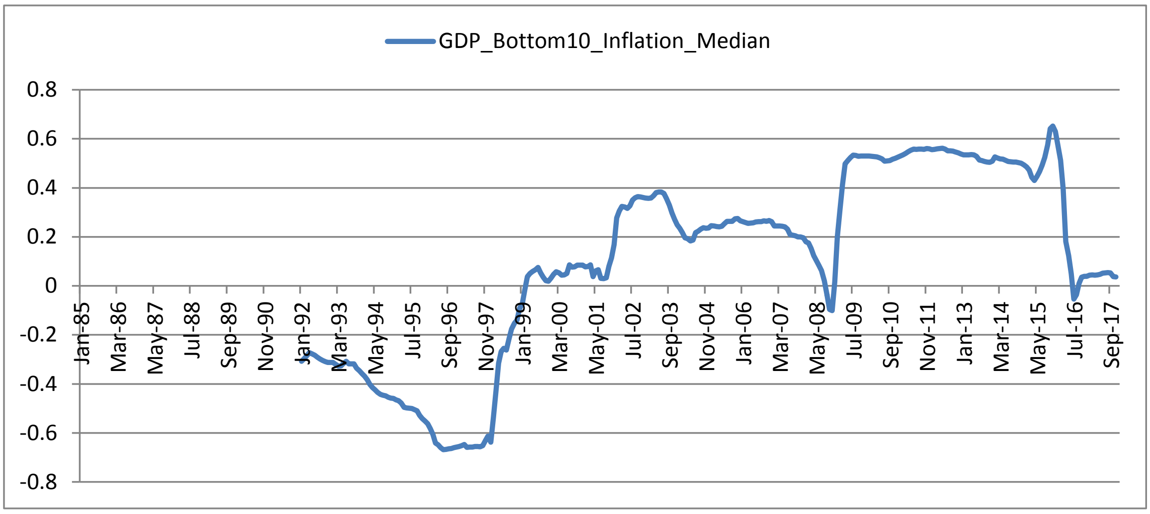


Figure 4: Correlation between worst-case GDP growth and inflation forecasts
Seven-year rolling window correlation between the bottom 10 average of individual GDP growth forecasts and median CPI inflation forecasts. Survey data are one-quarter-ahead forecasts from the BCFF and are monthly from 1985 to 2017.

suggested by Anderson, Hansen, and Sargent (2003) to provide a sense of the magnitude of the size of the ambiguity.

This approach quantifies the statistical closeness of two measures by calculating the average error probability in a Bayesian likelihood ratio test of two competing models. Intuitively, measures that are statistically close will be associated with large error probabilities, but measures that are easy to distinguish imply low error probabilities. Formally, let $l$ be the log likelihood function of the worst-case measure relative to the reference measure and $P^a$ be the alternative worst-case measure. Then, the average probability of a model detection error in the corresponding likelihood ratio test is $\epsilon = 0.5 \cdot P(l > 0) + 0.5 \cdot P^a(l < 0)$, where $\epsilon$ is just a simple equally weighted average of the probability of rejecting the reference model when it is true ($P(l > 0)$) and the probability of accepting the reference model when the worst-case model is true ($P^a(l < 0)$).

In general, a closed form expression for the detection error probability is not available. The error probability is calculated using simulated data. In this paper, parameters are estimated from data and the error detection probabilities for both output and inflation are at least 5%.

## 5. Conclusion

During the past decade, the joint dynamics of output growth expectations and inflation expectations have changed, and it is difficult to reconcile the behavior of the term structure of the nominal bond yield curve with leading equilibrium asset pricing models. Moreover, the long enough history of TIPS yield curves in the US suggests that the real yield curve is an important dimension to consider for equilibrium models. Recent studies that revisit the expectations hypothesis using survey expectations call for a new theory to understand the source of bond return predictability. It is also important for an equilibrium model of bond pricing to capture recent empirical findings on dividend or equity yields.

This is the first paper that provides an equilibrium model that is consistent with all the above evidence. Departing from the rational expectation hypothesis, we assume that the investor is ambiguity averse and evaluates future prospects under the worst-case measure (his subjective equilibrium belief). In the data, the term structure of ambiguity for inflation is upward sloping before the late 1990s, and slopes downward afterwards; the ambiguity yield curve for real output growth is always downward sloping. The ambiguity yields are linked with bond yields and equity yields through the recursive multiple priors preference in equilibrium.

For both subperiods, the worst-case distribution for output growth is the lower bound of the set of alternative mean growth rates, which are upward sloping because of the downward-sloping output forecast dispersion. Thus the real bond yield is always upward sloping. Before the late 1990s, when inflation expectation is negatively associated with the worst-case growth expectation, ambiguity averse investors pick the upper bound from the set of alternative mean inflation scenarios, which is upward sloping. This generates an upward-sloping nominal yield curve. During the second subperiod, inflation expectation is positively associated with the worst-case growth expectation, and the worst-case mean inflation is the lower bound now. However, at the same time, the inflation forecast dispersion turns to be downward sloping, which again implies an upward-sloping mean inflation in equilibrium. Therefore the model generates upward-sloping nominal yield curves in both subperiods, but with a different mechanism.

Realized ambiguity contains no trends because the true expectation of GDP growth and inflation evolves under the reference distribution, hence the realized short rates are lower than investors expected under their worst-case belief. Both this difference and current yield spreads/forward rates are driven by the trend components in the ambiguity process, which implies that the realized excess bond returns are predictable.

This model is also consistent with empirical findings on equity yields that the dividend future returns are slightly upward sloping and the volatility of equity yields is downward sloping. And the market returns are not significantly different from individual dividend spot returns.

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*Appendix.1. Forcing process*

Under the worst-case measure, the economic dynamics follow

$$
\begin{aligned}
z_{t+1} &= \phi_a a_t + \mu_z + x_{z,t} + \sigma^z \varepsilon_{t+1} \\
x_{z,t+1} &= \rho_x x_{z,t} + \sigma^x \varepsilon_{t+1} \\
\Delta d_{t+1} &= \zeta_d \Delta g_{t+1} + \mu_d + \sigma_d \varepsilon_{d,t+1} \\
a_{t+1} &= \mu_a + a_t + \sigma^a \varepsilon^a_{t+1}
\end{aligned}
$$

where $z_{t+1} = (\Delta g_{t+1}, \pi_{t+1})^T$, $x_{z,t+1} = (x_{c,t+1},\, x_{\pi,t+1})^T$, $a_{t+1} = (a_{c,t+1},\, a_{\pi,t+1})^T$, $\mu_z = (\mu_c,\, \mu_\pi)^T$, $\mu_a = (\mu^a_c,\, \mu^a_\pi)^T$, $\rho_x = \begin{pmatrix} \rho_c & 0 \\ 0 & \rho_\pi \end{pmatrix}$, $\phi_a = \begin{pmatrix} \phi^a_c & 0 \\ 0 & \phi^a_\pi \end{pmatrix}$, $\sigma^z = \begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_\pi \end{pmatrix}$, $\sigma^x = \begin{pmatrix} \sigma^x_c & \sigma^x_{c\pi} \\ 0 & \sigma^x_\pi \end{pmatrix}$, $\sigma^a = \begin{pmatrix} \sigma_{ac} & \sigma^{ac}_a \\ 0 & \sigma^{a\pi}_a \end{pmatrix}$, $\varepsilon_{t+1} = (\varepsilon_{c,t+1}, \varepsilon_{\pi,t+1})^T$, and $\varepsilon^a_{t+1} = (\varepsilon_{ac,t+1}, \varepsilon_{a,t+1})^T$. The shocks $\varepsilon_{c,t+1}$, $\varepsilon_{\pi,t+1}$, $\varepsilon_{d,t+1}$, $\varepsilon_{ac,t+1}$, and $\varepsilon_{a,t+1} \sim i.i.d.\ N(0,1)$. $\phi^a_c$ and $\phi^a_c$ represent the equilibrium choice of the upper or lower bound, equal to $-1$ or $+1$.

*Appendix.2. Stochastic discount factor*

Given the CRRA utility, the nominal stochastic discount factor can be written as

$$
m^{\$}_{t,t+1} = log\beta - \gamma \Delta g_{t+1} - \pi_{c,t+1} = log\beta - v' z_{t+1}
$$

where $v' = (\gamma,\, 1)$. For the real stochastic discount factor, we can replace $v'$ with $v' = (\gamma,\, 0)$.

*Appendix.3. Bond yields*

The time-$t$ price of a zero-coupon bond that pays one unit of consumption $n$ periods from now is denoted $P^{(n)}_t$, and it satisfies the recursion

$$P_t^{(n)} = E_{p_t^o}[M_{t,t+1}^{\$} P_{t+1}^{(n-1)}]$$

with the initial condition that $P_t^{(0)} = 1$ and $E_{p_t^o}$ is the expectation operator for the worst-case measure. Given the linear Gaussian framework, we assume that $p_t^{(n)} = log(P_t^{(n)})$ is a linear function of $a_t$ and $x_t$:

$$p_t^{(n)} = -A^{(n)} - B^{(n)} x_t - C^{(n)} a_t.$$

When we substitute $p_t^{(n)}$ and $p_{t+1}^{(n-1)}$ in the Euler equation, the solution coefficients in the pricing equation can be solved with $B^{(n)} = B^{(n-1)} \rho_x + v' = v' \left( \sum_{i=o}^{n-1} (\rho_x)^i \right)$, $C^{(n)} = C^{(n-1)} + v' \phi_a = v' \phi_a n$, and

$$A^{(n)} = \begin{array}{c} A^{(n-1)} - log\beta + v'\mu_z + C^{(n-1)}\mu_a - B^{(n-1)}\sigma^x \sigma^{z'} v \\ -0.5 * \left( v'\sigma^z \sigma^{z'} v + B^{(n-1)} \sigma^x \sigma^{x'} B^{(n-1)'} + C^{(n-1)} \sigma^a \sigma^{a'} C^{(n-1)'} \right) \end{array}$$

Nominal bond yields can be calculated as $y_t^{(n)} = -\frac{1}{n} p_t^{(n)} = \frac{A^{(n)}}{n} + \frac{B^{(n)}}{n} x_t + \frac{C^{(n)}}{n} a_t$. The log holding period return from buying an $n$ periods bond at time $t$ and selling it as an $n-1$ periods bond at time $t-1$ is defined as $r_{n,t+1} = p_{t+1}^{(n-1)} - p_t^{(n)}$, and the subjective excess return is $er_{n,t+1} = -Cov_t \left( r_{n,t+1}, m_{t,t+1}^{\$} \right) = -B^{(n-1)} \sigma^x \sigma^{z'} v$. The yield volatility is calculated as

$$\begin{aligned} Var_t \left( y_t^{(n)} \right) &= \left( \frac{B^{(n)}}{n} \sigma^x \right) \left( \frac{B^{(n)}}{n} \sigma^x \right)' \\ &+ \left( \frac{B^{(n)}}{n} \rho_x \sigma^x \right) \left( \frac{B^{(n)}}{n} \rho_x \sigma^x \right)' \\ &+ \ ... \\ &+ \left( \frac{B^{(n)}}{n} \rho_x^{t-1} \sigma^x \right) \left( \frac{B^{(n)}}{n} \rho_x^{t-1} \sigma^x \right)' \\ &+ \ t \left( \frac{C^{(n)}}{n} \sigma^a \right) \left( \frac{C^{(n)}}{n} \sigma^a \right)' \end{aligned}$$

To solve the price and yields for real bonds, we can just replace $\upsilon'$ with $\upsilon' = (\gamma, 0)$.

*Appendix.4. Expectations hypothesis*

To derive implications for the test in equation (13), let $A \equiv y_t^n - y_t^1$ and $B \equiv (n-1)\left(y_{t+1}^{n-1} - y_t^n\right)$. Since all shocks are Gaussian and orthogonal, they can be thought of as the error term in equation (13). The derivation in this session will ignore all shocks. Given the solution for bond yields, we can solve $A$ and $B$ as

$$\begin{aligned}
A &= \frac{A^{(n)}}{n} - A^{(1)} + \left(\frac{B^{(n)}}{n} - B^{(1)}\right) x_t \\
B &= A + VarCov(n-1) + C^{(n-1)}\left(a_{t+1} - a_t - \mu_a\right) \\
VarCov(n-1) &= 0.5 * \left(B^{(n-1)}\sigma^x\sigma^{x'}B^{(n-1)'} + C^{(n-1)}\sigma^a\sigma^{a'}C^{(n-1)'}\right) + B^{(n-1)}\sigma^x\sigma^{z'}\upsilon.
\end{aligned}$$

So the difference between $A$ and $B$ is $VarCov(n-1) + C^{(n-1)}\left(a_{t+1} - a_t - \mu_a\right)$. $VarCov(n-1)$ is quantitatively very small; the difference is mainly driven by $C^{(n-1)}\left(a_{t+1} - a_t - \mu_a\right)$.

When evaluating future prospects, investors' worst-case beliefs are described by $a_{t+1} = \mu_a + a_t + \sigma^a\varepsilon_{t+1}^a$. The difference between $A$ and $B$ now only contains the variance and covariance term $VarCov(n-1)$, which is very small. Thus the expectations hypothesis roughly holds.

However, the realized ambiguity process is described by $a_{t+1} = a_t + \sigma^a\varepsilon_{t+1}^a$, and now the difference is $VarCov(n-1) - \mu_a C^{(n-1)}$. To see intuitively what this difference implies for the expectations hypothesis test coefficient $\beta_n$, we first ignore the $x_t$ in $A$ and $B$, and then calculate $A$ and $B$ for different horizons. For $n = 2$:

$$\begin{aligned}
A &= \frac{1}{2}\mu_a C^{(1)} - \frac{1}{2}VarCov(1) \\
B &= -A \\
\beta_2 &\approx -1.
\end{aligned}$$

For $n = 3$:

$$
\begin{aligned}
A &= \mu_a C^{(1)} - \frac{1}{3}\left(VarCov(2) + VarCov(1)\right) \\
B &= -A + \frac{1}{3}VarCov(2) - \frac{2}{3}VarCov(1) \\
\beta_2 &\approx -1.
\end{aligned}
$$

For $n = 4$:

$$
\begin{aligned}
A &= \frac{3}{2}\mu_a C^{(1)} - \frac{1}{4}\left(VarCov(3) + VarCov(2) + VarCov(1)\right) \\
B &= -A + \frac{1}{2}\left(VarCov(3) - VarCov(2) - VarCov(1)\right) \\
\beta_2 &\approx -1.
\end{aligned}
$$

Similarly, we can calculate $\beta_n$ for $n = 5, 6, 7...$ If we ignore the variance/covariance term and $x_t$, the coefficient $\beta_n = -1$ for all $n$. To see the exact value for $\beta_n$, we should use simulated values for $x_t$, and take into account $VarCov(n-1)$.

*Appendix.5. Equity yields and returns*

Equity price and returns can be solved using the real stochastic discount factor $m_{t,t+1} = log\beta - \gamma\Delta g_{t+1}$. For any asset $j$ with a real payoff, the first-order condition yields the following asset pricing Euler condition:

$$E_{p_t^o}[exp(m_{t,t+1} + r_{j,t+1})] = 1$$

where $E_{p_t^o}$ is the expectation operator for the worst-case measure, and $r_{j,t+1}$ is the log of the gross return on asset $j$.

To solve the market return, it is assumed that the log price-dividend ratio for dividend claims, $z_t$, is linear in $a_{c,t}$ and $x_{c,t}$:

$$z_t = A_0 + A_1 x_{c,t} + A_2 a_{c,t}.$$

The log market return is given by the Campbell and Shiller approximation

$$r_{m,t+1} = k_0 + k_1 z_{t+1} + \Delta d_{t+1} - z_t$$

where $k_0$ and $k_1$ are log linearization constants. As noticed by previous studies,[21] the parameters $A_0$ and $A_1$ determine the mean of the price-consumption ratio, $\bar{z}$, and the parameters $k_0$ and $k_1$ are nonlinear functions of $\bar{z}$ with $\bar{z} = A_0(\bar{z}) + A_1(\bar{z})a$. $k_0$ and $k_1$ are given by $k_0 = log(1+exp(\bar{z})) - \bar{z}k_1$, $k_1 = \frac{exp(\bar{z})}{1+exp(\bar{z})}$. To get a highly accurate approximation, we need to iterate numerically until a fixed point for $\bar{z}$ is found.

By substituting $r_{m,t+1}$ and $z_t$ into the Euler equation, we can solve $A_0$, $A_1$, and $A_2$ with $A_1 = \frac{\zeta_d - \gamma}{1-k_1\rho_c}$, $A_2 = -\frac{\zeta_d-\gamma}{1-k_1}$, and

$$A_0 = \frac{\begin{array}{c} log\beta + k_0 + (\zeta_d - \gamma)\mu_c + \mu_d + k_1 A_2 \mu_c^a \\ 0.5\left((\zeta_d - \gamma)\sigma_c + k_1 A_1 \sigma_c^x \sigma_c\right)^2 + 0.5\left(k_1 A_1 \sigma_{c\pi}^x \sigma_\pi\right)^2 \\ +0.5\left(k_1 A_2 \sigma_{ac}\right)^2 + 0.5\left(k_1 A_2 \sigma_a^{ac}\right)^2 + 0.5\sigma_d^2 \end{array}}{1-k_1}$$

Given $A_0$, $A_1$, and $A_2$, the coefficients for expected returns under the reference measure $\mathbb{E}_t(r_{m,t+1}) = A_{0,E} + A_{1,E}x_{c,t} + A_{2,E}a_{c,t}$ can be calculated as $A_1^E = \gamma$, $A_2^E = \zeta_d - \gamma$, and $A_0^E = k_0 + (k_1 - 1)A_0 + \mu_d + \zeta_d\mu_c + k_1 A_2 \mu_c^a$.

For the price of individual dividends (or dividend strips), we can solve it in a similar way. Let $P_{t,n}$ denote the price of a dividend at time $t$ that is paid $n$ periods in the future. Let $D_{t+1}$ denote the realized dividend in period $t+1$. The price of the first dividend strip is given by $P_{t,1} = E_{p_t^o}[M_{t,t+1}D_{t+1}] = D_t E_{p_t^o}[M_{t,t+1}\frac{D_{t+1}}{D_t}]$, and the recursion $P_{t,n} = E_{p_t^o}[M_{t,t+1}P_{t+1,n-1}]$ allows us to compute the remaining dividend strip prices. Given the linear Gaussian framework, we assume that the log dividend strip prices, scaled by the current dividend, are also affine in the state variables:

$$pd_t^{(n)} = A_0^{(n)} + A_1^n x_{c,t} + A_2^{(n)} a_{c,t}.$$

[21] Campbell (1993); Campbell and Koo (1997); Bansal, Kiku, and Yaron (2007); Beeler and Campbell (2012)

Similar to the bond prices, we can first compute $pd_t^{(1)}$ using $pd_t^{(1)} = log\left(E_{p_t^o}[M_{t,t+1}\frac{D_{t+1}}{D_t}]\right)$, and then use the recursion $pd_t^{(n)} = log\left(E_{p_t^o}[exp\left(m_{t,t+1} + \Delta d_{t+1} + pd_{t+1}^{(n-1)}\right)]\right)$ to compute the remaining dividend strip prices. The solution coefficients in the pricing equation are $A_1^n = A_1^{n-1}\rho_c + \zeta_d - \gamma = (\zeta_d - \gamma)\left(\sum_{i=o}^{n-1}(\rho_c)^i\right)$, $A_2^{(n)} = A_2^{(n-1)} - (\zeta_d - \gamma) = -n(\zeta_d - \gamma)$, and

$$A_0^{(n)} = \begin{array}{c} A_0^{(1)} + A_0^{(n-1)} + A_2^{(n-1)}\mu_c^a + 0.5\left(A_1^{(n-1)}\right)^2((\sigma_c^x)^2 + (\sigma_{c\pi}^x)^2) \\ +0.5\left(A_2^{(n-1)}\right)^2((\sigma_{ac})^2 + (\sigma_a^{ac})^2) + (\zeta_d - \gamma)A_1^{(n-1)}\sigma_c^x\sigma_c \end{array}$$

Dividend spot yield or equity spot yield is defined as $ey_t^n = -\frac{1}{n}pd_t^{(n)}$, and dividend future yield is defined as dividend spot yield less the real bond yield of the same maturity. Dividend spot return is defined as $log(P_{t+1,n-1}) - log(P_{t,n}) = pd_{t+1}^{(n-1)} - pd_t^{(n)} + \Delta d_{t+1}$, and dividend future return is the dividend spot return less the same horizon bond holding period return. The volatility for dividend return and yield can be calculated given the closed form solutions in prices. Note that the expected returns are calculated under the reference measure.