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Central Bank Digital Currency and Transmission of Monetary Policy

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Abstract

How does the introduction of a central bank digital currency (CBDC) affect the transmission of monetary policy? Is paying interest on central bank liabilities contractionary or expansionary? Do CBDC design features—such as its substitutability with bank deposits and whether it bears interest—matter for monetary policy transmission? To address these questions, we develop a general equilibrium model with nominal rigidities, liquidity frictions, and a banking sector in which commercial banks face a leverage constraint. In the model, both CBDC and commercial bank deposits provide liquidity services to households. Banks issue deposits, extend loans to firms, and back these deposits with loans and central bank reserves. We find that an expansionary reserve quantity shock causes disinvestment, that is, higher reserves crowd out private investment. The economy's response to an increase in the interest rate on CBDC depends on the monetary policy framework: such a change is expansionary when policy targets the reserve interest rate and contractionary when it targets the CBDC interest rate. Moreover, the response to a change in the interest rate on reserves depends on the central bank's balance sheet management—specifically, how the quantities of CBDC and reserves are determined. Finally, CBDC design matters: imperfect substitutability between a zero-interest CBDC and deposits amplifies macroeconomic responses to monetary policy shocks, while high substitutability yields outcomes that are essentially the same as those in an economy without a CBDC.

Topics: Digital currencies and fintech, Interest rates, Monetary policy, Monetary policy framework, Monetary policy transmission, Monetary aggregates
JEL codes: E31, E4, E58, G21, G51

Résumé

Comment l'introduction d'une monnaie numérique de banque centrale (MNBC) influence-t-elle la transmission de la politique monétaire? Le paiement d'intérêts sur le passif de la banque centrale a-t-il un effet restrictif ou expansionniste? Les caractéristiques de conception d'une MNBC – telles que sa fongibilité avec les dépôts bancaires et la possibilité de lui faire produire ou non des intérêts – ont-elles de l'importance pour la transmission de la politique monétaire? Pour répondre à ces questions, nous élaborons un modèle d'équilibre général comportant des rigidités nominales, des frictions relatives à la liquidité et un secteur bancaire dans lequel les banques commerciales sont assujetties à une contrainte de levier d'endettement. Dans ce modèle, la MNBC et les dépôts dans les banques commerciales peuvent tous les deux être utilisés comme modes de paiement et constituent un vecteur de liquidité pour les ménages. Les banques offrent des services de dépôt, octroient des prêts aux entreprises et garantissent ces dépôts par des prêts et les réserves de banque centrale. Nous constatons qu'un choc lié à l'expansion des réserves entraîne un désinvestissement, c'est-à-dire que des réserves plus importantes ont un effet d'éviction sur les investissements privés. La réaction de l'économie à une hausse du taux d'intérêt de la MNBC dépend du cadre de conduite de la politique monétaire : un tel changement est expansionniste lorsque la politique monétaire cible le taux d’intérêt de réserve et restrictif lorsqu’elle cible le taux d’intérêt de la MNBC. En outre, la réaction à une variation du taux d’intérêt sur les réserves est influencée par la gestion du bilan de la banque centrale et, plus précisément, la façon dont les quantités de MNBC et de réserves sont déterminées. Enfin, la conception de la MNBC est importante : la fongibilité imparfaite entre une MNBC ne portant pas intérêt et les dépôts amplifie les réponses macroéconomiques aux chocs de politique monétaire, tandis qu’une fongibilité élevée donne des résultats essentiellement identiques à ceux d’une économie sans MNBC.

Sujets : Monnaies numériques et technologies financières; Taux d’intérêt; Politique monétaire; Cadre de la politique monétaire; Transmission de la politique monétaire; Agrégats
Codes JEL : E31, E4, E58, G21, G51

1 Introduction

The effects of monetary policy (MP) shocks on the real economy remain a central question in macroeconomics. The dominant analytical framework for studying the MP transmission, the standard new Keynesian (NK) model that features nominal rigidities, while highly useful for policy analysis, exhibits two notable limitations. First, money is absent from most NK models: transactions and liquidity services play no explicit role in shaping equilibrium outcomes and the quantity of money is largely irrelevant for real variables. Second, the policy rate set by the central bank is usually assumed to be the same interest rate that is relevant for households and firms. This abstraction removes the role of banks and the distinction between different interest rates in the economy.

At the same time, recent developments in payment technologies—particularly the prospect of introduction of a central bank digital currency (CBDC) in many countries—have renewed interest in how the MP transmission may change in the presence of a publicly provided, potentially interest-bearing, liquid asset that competes with commercial bank deposits. As a means of payment, a CBDC would compete with bank deposits and thereby have implications for the banking and financial system. As a store of value, CBDC would be used along with bank deposits, government bonds, and other safe assets, which would have macroeconomics implications. Altogether, a CBDC could alter the relative demand for bank deposits, the supply of credit, and the equilibrium determination of interest rates, thereby altering the effectiveness of traditional MP tools.

These considerations call for a framework in which money, banking, and CBDC can be jointly analyzed within a coherent model of MP transmission. In this paper, we develop such a framework by embedding money and banking into an NK model. In the model, banks subject to financial frictions issue deposits that could compete with CBDC as instruments providing liquidity, with two key extensions. First, we incorporate complementarity between consumption and real balances, a property inherited from New Monetarist (NM) models. Second, we allow for rich CBDC design features, including varying degrees of substitutability between CBDC and bank deposits.

Banks play an important role in our model. On the liability side, they issue deposits that are used to buy final goods. On the asset side, they hold reserves and invest in loans that are used in the process of producing final goods. Therefore, banks influence both the demand and supply sides of the economy through issuing reserves and investing in loans, respectively. Traditional monetary policy shocks (to the rate on or quantity of reserves) affect banks' behavior through the leverage constraint. Novel monetary shocks (on the rate on or quantity of CBDC) affect household demand for deposits, which in turn affects banks' optimal decision that governs allocation of their assets and liabilities.

By taking liquidity and financial frictions seriously, our model bridges the gap between NK and NM approaches. Our model allows for separate policy instruments—a short-term interest rate on reserves and the quantity of reserves—and examines their effects on the macroeconomy both with and without a CBDC.[1]

To understand the basic economic forces in the model, we first study the impact of a canonical monetary policy shock, an increase in the interest rate on central bank reserves, in the absence of a CBDC. There are three main channels through which the shock transmits to the economy. Firstly, a standard NK channel, where an increase in illiquid bond interest rates leads to lower current consumption via the Euler equation and thus reduced aggregate demand and output, which in turn impacts inflation via the NK Phillips curve that arises due to sticky prices. Secondly, a New Monetarist (NM) channel, where a narrower spread between illiquid bond and bank deposit rates reduces the opportunity cost of holding deposits (cost of liquidity), thereby boosting deposit demand and consumption, as well as labor supply and output in the presence of complementarity. Lastly, a supply channel, where higher capital costs arising from higher costs of bank funding reduces investment, leading to a decline in output. These three channels together illustrate the interplay of economic forces following a monetary policy shock in the model.

Our first novel result is that an expansionary reserves quantity shock (similar to a money supply shock in the old monetary literature) can lead to disinvestment—a decline in investment despite higher reserves—regardless of whether there is complementarity between consumption and real balances or whether a CBDC is present. That is, an abundance of reserves crowds out private investment. Before we explain the economic mechanism behind this result, we want to clarify the distinction between disinvestment and disintermediation. In many papers in the CBDC literature, disintermediation generally refers to the situation in which banks issue fewer deposits, like in Keister and Sanches (2023) and Chiu et al. (2023). Our result, however, is about disinvestment, where real investment declines, but deposits issued by banks in fact increases. That is, as the quantity of reserves increases in the system, the banks issue more deposits (so more intermediation, not disintermediation) but invest in fewer loans, thus resulting in less private investment.

The mechanism underlying our disinvestment result is as follows. When the central bank injects

[1]We extend the model in the appendix to include fiat cash. In the main text, we do not include cash to make the exposition simpler.

additional reserves into the system, the spread on central bank reserves, defined as the rate on illiquid bonds minus the interest paid on reserves, must decrease to provide banks with the incentive to hold these reserves. As long as central bank reserves and claims to capital are (to some extent) substitutes in satisfying the leverage constraint, a reduction in the spread on reserves translates into a reduction in the spread on capital returns. Holding the interest rate on illiquid bonds constant, this implies that the rate of return required on capital must increase. Consequently, capital becomes more costly, leading to a decline in investment. Interestingly, this mechanism operates whether or not there is complementarity between consumption and money.

Two elements are central to this disinvestment result. First, deposits carry a liquidity premium, and reserves inherit this premium because they provide backing for deposits. Second, reserves and capital claims are substitutes in meeting banks' leverage requirements—albeit with reserves considered higher-quality assets—so that changes in the interest rate on reserves are transmitted directly to the cost of capital.

Results in the literature that might appear puzzling when viewed solely through a single theoretical lens can be reconciled within our framework. For example, in NM models, paying interest on money is typically expansionary, whereas in NK models—and in empirical findings—higher policy rates are generally considered contractionary. Our framework reconciles these perspectives by distinguishing payment instruments, the set of agents allowed to hold them, and the channel through which central bank policy is implemented. In our setup, only banks hold reserves, only households hold CBDC, deposits and CBDC provide liquidity services with potentially imperfect substitutability, and monetary policy can be conducted via either CBDC or reserves. This structure allows us to examine how alternative policy regimes shape the effects of interest rate changes, leading directly to our next result.

Our second novel result is that the monetary policy framework itself critically shapes the effects of interest rate changes: a CBDC interest rate shock has opposite effects on consumption, output, investment and inflation depending on whether monetary policy is conducted through reserves or through CBDC. To illustrate this, we compare responses to the same shock—a change in the CBDC interest rate—under two policy regimes: (i) a Taylor rule in which it sets the reserves interest rate, and (ii) a Taylor rule in which it sets the CBDC interest rate. Focusing on a positive CBDC interest rate shock, we find that when reserves are the primary policy instrument, a higher CBDC interest rate becomes expansionary: by reducing the opportunity cost of liquidity for households, it increases labor supply and consumption, operating through a channel familiar from NM models. In contrast, when CBDC is the primary policy instrument, the shock is contractionary, resembling the standard NK transmission mechanism for an interest rate shock. This finding underscores that the choice of policy instrument and framework—not just the level and change in interest rates—can fundamentally alter macroeconomic outcomes.

We then turn to perhaps a more practical part of the paper and study effects of a zero-interest CBDC—the form most frequently discussed by central banks. Our third novel result concerns the role of CBDC design in shaping macroeconomic dynamics. CBDC design matters in a non-trivial way: When the CBDC is an *imperfect* substitute for deposits, it amplifies the responses of consumption, output, and inflation to monetary policy shocks (thus for both shocks to interest rate on reserves or to quantity of reserves). When the CBDC is a *perfect* substitute with deposits, the responses to shocks are essentially the same as those observed in the model without a CBDC. This finding highlights that the substitutability between CBDC and deposits is a key parameter governing the transmission of monetary policy.

Finally, we show that the response of the economy to a standard monetary policy shock depends on balance sheet quantity rules that govern the evolution of the central bank balance sheet variables. To best illustrate this point, we compare two scenarios, one in which the central bank fixes the CBDC rate and one in which it fixes the CBDC quantity. In response to a *standard* reserves interest rate shock (i.e., a shock in the Taylor rule that governs the interest rate on reserves), fixing the CBDC interest rate leads to a significant fall in output and consumption compared to fixing the CBDC quantity. What explains this result? When the CBDC interest rate is fixed, the opportunity cost of holding CBDC rises with an increase in the illiquid bond rate, making liquidity more expensive and reducing consumption and output further. When the CBDC quantity is fixed and its interest rate is flexible, the CBDC rate increases in response to pressure from rising deposit rates, activating the NM channel and mitigating contractionary effects of the shock. This exercise thus highlights the importance of the balance sheet quantity rules in determining MP transmission, which is in stark contrast to standard NK models where changes in the quantity of real money balances have no independent effect on other real macroeconomic variables once the short-term nominal rate is determined.

Literature. Our paper is, on the one hand, related to standard macro models that model financial frictions explicitly. Some related models include Gertler and Karadi (2011), Gertler and Karadi (2015), Gertler and Kiyotaki (2010), Sims and Wu (2021), Benigno and Benigno (2021), and Bhattarai and Neely (2022). On the other hand, it is related to the growing literature on CBDCs, covering a wide range of topics. In this part, we first compare our results with the most closely related papers including Piazzesi et al. (2019), regarding the transmission of monetary policy shocks, and Chiu et al. (2023), regarding the main economic forces at play in the steady state. Next, we discuss the related literature more broadly.

Piazzesi et al. (2019) study an NK model with money-in-the-utility function and complementarity between consumption and money. There are two key differences between our model and theirs. First, in our model, the CBDC and bank deposits *both* provide liquidity to households and could be substitutes or complements depending on the design of the CBDC. In their paper, agents either use central bank money (Section 2) or bank deposits (Section 3) and the paper does not consider the effects of a CBDC on the banking system. Second, banks in our model lend to firms to finance their capital expenditure, whereas in Piazzesi et al. (2019), banks simply invest in some assets with an exogenous rate of return. Modeling the interactions between deposits and other means of payments provides insight into how changes in the interest rate of a CBDC or its design features affect demand for bank deposits (the first difference), which in turn changes the cost of funding for firms, eventually affecting the supply side of the economy (the second difference).

In Chiu and Davoodalhosseini (2023), banks issue deposits and extend loans to firms, and the CBDC and bank deposits compete in the sense that both can be used in a fraction of transactions. The similarity between their model and ours is that in both models, the supply and demand sides of the economy are connected through banks' decisions. However, there are many differences between the two papers. They do not study dynamic responses to shocks in a model with nominal rigidities and capital accumulation and focus only on the steady-state analysis. Moreover, our model allows for a wider range of design features between a CBDC and other payment methods. Finally, in our model, banks face financial frictions, which gives rise to demand for central bank reserves. As a result, monetary policy affects this economy via a richer set of policy tools (i.e., inflation rate, the CBDC interest rate and quantity, and the reserves rate and quantity) relative to their paper, in which there is no demand for central bank reserves and the only central bank policies are the inflation rate and the interest rate on the CBDC.

Abad et al. (2023) study the implications of a CBDC for macroeconomic variables, focusing on frictions in the interbank market. Similar to our paper, they have a money-in-the-utility function framework, but they do not consider the complementarity between consumption and money that activates the NM channel in our paper. They also do not study effects of different types of monetary policy shocks. In a macroeconomic model with bank deposit market power, Paul et al. (2024) study welfare implications of introducing a CBDC as well as the transitional dynamics following the introduction of a CBDC in a closed economy framework. Assenmacher et al. (2023) incorporate NK frictions in an NM model. They find that having a CBDC does not markedly change the model's response to macroeconomic shocks. In contrast to the last two aforementioned papers, we provide nuances for the response of the economy in the presence of a CBDC and show that details matter substantively, i.e., the elasticity of substitution between CBDC and deposits, the monetary policy framework, and the balance sheet quantity rules, for macroeconomic outcomes.

In an extension of Minesso et al. (2022), Assenmacher et al. (2024) include financial frictions and occasionally binding constraints in a two-country DSGE model. They find that the transition from a non-CBDC regime to one with a CBDC triggers an initial surge in demand for CBDC and money, leading to the displacement of bank deposits and a subsequent decline in investment, consumption, and output. In a related work, Bidder et al. (2024) show that CBDC along with a proper holding limit increases financial stability and welfare. These papers focus mostly on the transitional dynamics of introducing a CBDC and not so much on the transmission channels of standard monetary policy shocks or balance sheet quantity shocks in the presence of a CBDC.

Similar to our paper, Gelfer and Gibbs (2023) obtain a disinvestment result in a two-country macro model with financial intermediation following an asset purchase policy, though from a different channel. In their paper, disinvestment comes from an open economy setting and changes in the real exchange rate. In our paper, the result comes from changes in the premium on reserves which spills over to the rate of return on capital.[2]

The rest of the paper is organized as follows. Section 2 introduces the model and analyzes the steady state equilibrium. Section 3 studies the dynamic responses to shocks. We also prove analytical results in this section for some special cases of the model. Section 4 includes the quantitative results and impulse response functions to shocks. Section 5 concludes. There are several appendices that are organized as follows. Appendix A collects equilibrium conditions in the benchmark model. Appendix B provides proofs of Propositions 1 and 2. Appendix C provides proofs of Proposition 3 and Corollary 1. Appendix D discusses the role of different model ingredients. Appendix E discusses a robustness check. Appendix F extends the model to include cash. Appendix G collects some special cases of the model for illustration.

[2] We organize other related literature into four categories. The first category, CBDC and banking, examines the impact of CBDC on the traditional banking system: Andolfatto (2021), Keister and Sanches (2023), and Chiu et al. (2023), as well as Fernández-Villaverde et al. (2021), Benigno et al. (2022), Niepelt (2020), and Williamson (2022a,b). The second category, CBDC and monetary policy, examines the macroeconomic consequences of CBDCs in DSGE, NM, and other models: Barrdear and Kumhof (2022), Assenmacher et al. (2023), Davoodalhosseini (2022), and Benigno and Benigno (2021). The third category investigates CBDC and financial stability: Fernández-Villaverde et al. (2021), Williamson (2022a), and Chiu et al. (2020). The fourth and final category explores digital currency design and platforms: Chiu and Koeppl (2019), Brunnermeier and Payne (2022), and Cheng et al. (2024).

2 Model

We use a relatively simple model which consists of households, firms, banks and the central bank. The time is discrete, $t = 0, 1, ...$, and goes forever. Banks issue deposits to households and make loans to firms. Households use these deposits as payment means, in addition to the CBDC issued by the government. In our model, CBDC can be understood as central bank accounts accessible to households that earn interest (if the central bank so decides) and can be used for payments.[3] In this version of the model, we do not include cash to simplify the analysis and clarify the mechanisms in action. Finally, the central bank issues reserves solely available to banks to back deposits.

Banks in our model are similar to the banks in Piazzesi et al. (2019). However, a key difference is that we incorporate the credit channel of monetary policy. In our model, firms depend on banks for their capital funding, meaning that changes in the cost of loans (stemming from shifts in the cost of reserves or deposits for banks) impact the supply side of the economy. Thus, a feedback loop exists from bank credit to firms' investment in capital and overall aggregate supply. In contrast, the banks in Piazzesi et al. (2019) invest in assets that do not affect the supply side of the economy. The production side in our model follows standard NK models with financial frictions, e.g, Gertler and Karadi (2011) and Gertler and Karadi (2015).

2.1 Households

Households consume a final good for which they need to use means of payments, including deposits and CBDC. We use the money-in-the-utility function framework and obtain a demand function for means of payments.[4] Using this model, one can study the implications of different design features of a CBDC for macroeconomic outcomes.

[3] The CBDC we consider in this paper is a retail CBDC which can be held only by households. However, we do not take a stance on the technological aspects of CBDC. The verification of the identity of the CBDC holder may or may not be required. CBDC can even be token based. We do not discuss details here. See the literature review by Chapman et al. (2023) for further discussion on the design of a CBDC. More broadly, see Ahnert et al. (2022) and Davoodalhosseini and Rivadeneyra (2020) for literature reviews about CBDCs and e-monies.

[4] There are various ways in the literature to motivate the money demand function including: money-in-the-utility function (Sidrauski (1967)), cash-in-advance constraint (Lucas Jr (1982)), shopping time technology (Kimbrough (1986)), or a transactions-cost technology (Feenstra (1986)).

2.1.1 Household Preferences and Maximization Problem

Preferences at time t over consumption good, means of payments, and labor are given by

$$U\left(C_t, \frac{D_t}{P_t}, \frac{F_t}{P_t}, H_t\right) = \frac{1}{1-\frac{1}{\sigma}}\left(C_t^{1-\frac{1}{\eta}} + \omega_D \mathbf{Liq}_t^{1-\frac{1}{\eta}}\right)^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}} - \frac{\psi}{1+\varphi} H_t^{1+\varphi},$$

where $\mathbf{Liq}_t$ is a liquidity aggregator,

$$\mathbf{Liq}_t \equiv \left((D_t/P_t)^{1-\frac{1}{\upsilon}} + \frac{\omega_{FD}}{\omega_D}(F_t/P_t)^{1-\frac{1}{\upsilon}}\right)^{\frac{1}{1-\frac{1}{\upsilon}}},$$

and D_t and F_t denote the nominal deposit and CBDC balances that bear interest rates i_t^D and i_t^F, respectively; H_t denotes the units of labor supplied; and P_t is the price level. As in Piazzesi et al. (2019), the utility function features η, which denotes the intratemporal elasticity of substitution between consumption and the liquidity aggregator, and σ, which denotes the intertemporal elasticity of substitution between the consumption bundle today and tomorrow.

Two points are in order regarding this period utility function. First, as long as $\sigma = \eta$, the utility is separable in consumption and the liquidity aggregator. If $\sigma > \eta$ ($\sigma < \eta$), consumption and the liquidity aggregator are complements (substitutes). Second, there are two design features of a CBDC. These features are determined by the CBDC technology, and the government might be able to affect them. First, υ denotes the elasticity of substitution between deposits and the CBDC. As υ increases, deposits and the CBDC become better substitutes and remain substitutes as long as $\upsilon > \eta$. Second, the term $\frac{\omega_{FD}}{\omega_D}$ captures the relative transaction costs for households.[5]

A household's expected lifetime utility maximization problem at date 0 is given by

$$\max_{C_t, D_t, F_t, H_t, S_t} \quad \boldsymbol{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{D_t}{P_t}, \frac{F_t}{P_t}, H_t\right)$$

$$\begin{aligned} \text{s.t.} \quad & P_t C_t + D_t + F_t + S_t \\ = \quad & W_t H_t - T_t + \Pi_t + D_{t-1}\left(1 + i_{t-1}^D\right) + F_{t-1}\left(1 + i_{t-1}^F\right) + S_{t-1}\left(1 + i_{t-1}^S\right). \end{aligned}$$

The nominal flow budget constraint is standard. Households receive wage income, $W_t H_t$ and

[5] For example, when the CBDC and deposits are perfect substitutes ($\upsilon = \infty$), $\frac{\omega_{FD}}{\omega_D} = 1.01$ implies that CBDC is 1% more effective than deposits in payments. This specification for CBDC is new and nests several special CBDC designs that have been discussed in the literature. For example, a deposit-like CBDC (as in Chiu and Davoodalhosseini (2023) a perfect substitute with bank deposits: $\upsilon = \infty$. One can map the market shares and merchants' acceptability to different values of υ, ω_D, and ω_{FD} and then experiment with different designs of the CBDC.

profits of the firms, Π_t and pay taxes, T_t, to the government (which can be negative). Households can invest in deposits and CBDC which pay interest rates i_t^D and i_t^F, respectively. They can also invest in riskless private or public bonds, denoted by S_t, which pay a nominal interest rate i_t^S. In Appendix D, we discuss the role of different model ingredients, for example, what changes if households cannot hold illiquid bonds. The household is also subject to a standard No Ponzi game constraint.

2.1.2 Households' Optimality Conditions

Denote by λ_t the Lagrangian multiplier on the budget constraint. Households' optimality conditions are presented in detail in Appendix A. The optimality conditions for S and for each $J \in \{D, F\}$ yield

$$\frac{U_{J,t}}{P_t} = \lambda_t - \beta \boldsymbol{E}_t \lambda_{t+1} \left(1 + i_t^S - (i_t^S - i_t^J)\right) = \beta \boldsymbol{E}_t \lambda_{t+1} \left(i_t^S - i_t^J\right) = \lambda_t \frac{i_t^S - i_t^J}{1 + i_t^S}.$$

The money demand function for deposits and the CBDC are given by

$$\frac{i_t^S - i_t^D}{1 + i_t^S} \geq \omega_D \left(V_{D,t}^{\frac{\eta}{v}} V_{FD,t}^{1-\frac{\eta}{v}}\right)^{\frac{1}{\eta}} \text{ with equality if } D_t > 0, \tag{2.1}$$

$$\frac{i_t^S - i_t^F}{1 + i_t^S} \geq \omega_{FD} \left(V_{F,t}^{\frac{\eta}{v}} V_{FD,t}^{1-\frac{\eta}{v}}\right)^{\frac{1}{\eta}} \text{ with equality if } F_t > 0, \tag{2.2}$$

where

$$\text{Velocity}: \quad V_{J,t} \equiv \frac{P_t C_t}{J_t} \text{ for } J \in \{D, F\},$$

and

$$V_{FD,t} \equiv \left(V_{D,t}^{\frac{1}{v}-1} + \frac{\omega_{FD}}{\omega_D} V_{F,t}^{\frac{1}{v}-1}\right)^{\frac{1}{\frac{1}{v}-1}}. \tag{2.3}$$

The money demand functions above are a generalization of those in Piazzesi et al. (2019). In their paper, there is neither a competition between the means of payments nor a rich set of substitution/complementarity patterns between them. To understand the economic forces in play, we focus on Equation (2.2) as Equation (2.1) is similar.

The LHS of Equation (2.2) shows the marginal cost (i.e., the opportunity cost) of holding a unit of CBDC. By holding CBDC, households forgo the interest that they could get by holding riskless bonds, i_t^S. Instead, they receive interest on the CBDC, i_t^F. The RHS of Equation (2.2) shows the marginal benefit of holding a unit of CBDC, which comprises of the benefit of using CBDC in conjunction with deposits. It is equal to a product of two terms, the velocity of the CBDC, V_D, and a composite velocity, V_{FD}. Here, by velocity we mean the standard definition, i.e., how much consumption can be purchased using one unit of money in circulation.

As CBDC and deposits become better substitutes (υ increases), the weight of CBDC velocity reduces and the weight of composite velocity increases. In an extreme case where υ becomes very large, which means that CBDC and deposits become perfect substitutes, only the composite velocity matter, which simply means that only aggregate balances ($D_t + \frac{\omega_{FD}}{\omega_D} F_t$) matters, not individual balances in deposits or CBDC. In another extreme case where $\upsilon = \eta$, only individual velocities matter. That is, CBDC and deposits demands are independent.

Note that i_t^D and i_t^F can go below zero depending on the liquidity service they provide to households.[6] Finally, Equation (2.2) is independent of σ, implying that the demand functions derived here do not depend on whether the utility function is separable in its arguments or not.

The optimality condition for labor supply is given by

$$C_t^{\frac{1}{\sigma}} \psi H_t^{\varphi} = Q_t^{\frac{\eta}{\sigma}-1} \frac{W_t}{P_t}, \tag{2.4}$$

where

$$Q_t \equiv \left(1 + \omega_D V_{FD,t}^{\frac{1-\eta}{\eta}}\right)^{\frac{1}{1-\eta}}. \tag{2.5}$$

Equation (2.4) denotes the optimal labor supply condition stating that the marginal rate of substitution between consumption and labor supply is simply equal to the relative price of the two, which is equal to the adjusted real wage. Piazzesi et al. (2019) obtained a similar condition except that the definition of Q_t has been modified in our case.

For the case of separable utility ($\eta = \sigma$), Equation (2.4) is self-explanatory. In the more general case of non-separable utility ($\eta \neq \sigma$), the real wage has to be adjusted by another term depending on Q_t. To provide a more concrete explanation, consider the case where consumption and means of payments are complements ($\sigma > \eta$), the case that resembles closely a model with a cash-in-advance constraint.[7] As the nominal interest rate increases, the opportunity cost of

[6] Yet, the model can introduce an effective lower bound for deposits for a given interest rate on CDBC. That lower bounds is the highest rate under which there is no demand for deposits.

[7] Empirical studies also confirm that $\sigma > \eta$ is indeed the relevant case. For example, σ is set to be 1 in Gali (2008) and η is assumed to be around 0.2 to 0.5. See the related empirical findings in Section 2.5.4 of Walsh (2017) textbook.

holding money rises, which works effectively as a tax on consumption and consequently a subsidy on leisure. Therefore, the household reduces labor supply as the opportunity cost of holding money increases. (See Chapter 3 of Walsh (2017).) To see this effect clearly in our model, simply put $\upsilon = \eta$ as a special case, which captures the case that all means of payments have the same elasticity with respect to change in interest rates. As the nominal interest rate i_t^S rises, the velocities rise too (from the demand functions for means of payments), implying that Q must increase. Therefore, the supply of labor, H, should fall from Equation (2.4), all else being equal.[8]

The Euler equation for illiquid bonds can be written as

$$\beta \boldsymbol{E}_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^S\right) = 1. \tag{2.6}$$

The optimal choice of holdings of an illiquid bond requires that the rate of return on the illiquid bond discounted by the stochastic discount factor be equal to one. Again, Equation (2.6) is standard in the separable case. In the non-separable case, particularly when $\sigma > \eta$, as the nominal interest increases, the opportunity cost of holding money increases. Therefore, the agent discounts the future less (as mentioned above in the discussion about Q_t), leading to less consumption today. As the Euler equation for illiquid bonds, Equation (2.6), plays a key role in monetary policy rate shock transmission under sticky prices, a difference in model dynamics between the separable and non-separable cases (particularly when $\sigma > \eta$) can arise if Q_t gets affected significantly.

To put everything together regarding a household's optimality conditions: Equation (2.6) is a generalization of a standard Euler equation in a case where consumption and means of payments are complements. This equation determines the relationship between the nominal interest rate on illiquid bonds with the consumption of housholds. Equation (2.1) and (2.2) then relate the velocity of different means of payments with the wedge between the nominal rate on illiquid bonds and the nominal interest rate on that means of payment. This wedge arises because of the liquidity services that these means of payments provide to agents.

[8] This channel is referred to as the "cost channel" in the NK literature, although it is often given relatively little attention. In contrast, models that incorporate cash-in-advance constraints or follow the NM approach consider this channel central to their frameworks. For example, in NM models, an increase in the opportunity cost of means of payments leads buyers to bring less real balances to the decentralized market. This means that sellers work and produce less, and as a result, output falls.

2.1.3 Understanding Q

First, use Equations (2.1) and (2.2) to write $V_{D,t}$ and $V_{F,t}$ as functions of $V_{FD,t}$ (assuming that D_t and F_t are both used in equilibrium), then use Equation (2.3) to solve for $V_{FD,t}$:

$$V_{FD,t} = \omega_D^{\frac{\eta}{\upsilon-1}} \left(\omega_D^{\upsilon} \left(\frac{i_t^S - i_t^D}{1 + i_t^S} \right)^{1-\upsilon} + \omega_{FD}^{\upsilon} \left(\frac{i_t^S - i_t^F}{1 + i_t^S} \right)^{1-\upsilon} \right)^{-\frac{\eta}{\upsilon-1}}. \tag{2.7}$$

Using Equation (2.1) again, we obtain

$$V_{D,t}^{\frac{1-\upsilon}{\upsilon}} = \omega_D^{\upsilon-\frac{\eta}{\upsilon}} \frac{\left(\frac{i_t^S - i_t^D}{1+i_t^S} \right)^{1-\upsilon}}{\left(\omega_D^{\upsilon} \left(\frac{i_t^S - i_t^D}{1+i_t^S} \right)^{1-\upsilon} + \omega_{FD}^{\upsilon} \left(\frac{i_t^S - i_t^F}{1+i_t^S} \right)^{1-\upsilon} \right)^{\frac{\upsilon-\eta}{\upsilon}}}. \tag{2.8}$$

Equation (2.8) is a modified demand function for deposits in the presence of a CBDC. In the case that a CBDC does not exist ($\omega_{FD} = 0$), it reduces to $V_{D,t} = \omega_D^{-\eta} \left(\left(i_t^S - i_t^D \right) / \left(1 + i_t^S \right) \right)^{\eta}$. Like a typical money demand equation, Equation (2.8) states that the velocity of deposits depends on the opportunity cost of deposits (the numerator), but this opportunity cost should be adjusted by an aggregator, which depends on the opportunity costs of both deposits and the CDBC (the denominator). For example, when the CBDC and deposits are perfect substitutes ($\upsilon \to \infty$), then the velocity of deposits would be too large (i.e., there would be no demand for deposits) if $i_t^F > i_t^D$ while it would be unaffected by CBDC if $i_t^F < i_t^D$.

Second, we calculate Q_t using Equation (2.7):

$$Q_t = \left(1 + \omega_D^{\frac{\upsilon-\eta}{\upsilon-1}} \left(\omega_D^{\upsilon} \left(\frac{i_t^S - i_t^D}{1 + i_t^S} \right)^{1-\upsilon} + \omega_{FD}^{\upsilon} \left(\frac{i_t^S - i_t^F}{1 + i_t^S} \right)^{1-\upsilon} \right)^{-\frac{1-\eta}{\upsilon-1}} \right)^{\frac{1}{1-\eta}}. \tag{2.9}$$

Equation (2.9) shows how Q_t depends on two interest rate spreads in our model. Given our discussion above about the role of Q_t as an additional variable in the optimal labor supply and Euler equations (Equations (2.4) and 2.6)) for the non-separable utility function case, it then points out how the key channels of monetary policy transmission can change compared to the standard models in the literature (which are represented by the separable utility function version of our model) once interest rate spreads evolve non-trivially over time.

2.1.4 The Role of CBDC

There are two roles of CBDC in our model. First, it changes the demand for deposits, and the change depends on different parameters including the elasticity of substitution between deposits and the CBDC (through interactions of Equations (2.1) and (2.2)).

Second, it provides liquidity to households just like deposits (see how the CBDC interest rate affects Equation (2.9)), so it changes the incentives to supply labor and to consume through optimal labor supply and Euler equations. In Equation (2.9), a higher opportunity cost of the CBDC increases Q_t. A higher Q_t implies that the adjusted real wage in Equation (2.4) falls, leading to lower consumption. The magnitude, not the sign, of the effect depends on both the elasticity of substitution between deposits and the CBDC (υ) and the intratemporal substitution between consumption and the liquidity aggregator (η). Moreover, a higher Q_t, everything else equal, leads to a lower consumption through Equation (2.6) when $\sigma > \eta$.

2.2 Banks

Banks are short-lived; they are born at the beginning of period t and die at the end of period $t+1$. They are perfectly competitive.[9] On the liability side, they can issue liquid deposits, D_t, which are used in transactions by households as we explained previously while presenting the household's problem. They can also issue illiquid bonds, A_t, which are perfectly safe and cost less to adjust. We note that A_t can be negative. On the asset side, they can buy reserves from the central bank, M_t, or they can invest in real claims on physical capital, b_t. The interest rate on reserves is denoted by i_t^M, and the interest on deposits is equal to i_t^D. The cost of issuing illiquid bonds is equal to i_t^S, which is the nominal interest rate on government bonds too.[10]

2.2.1 Constraints and Bank's Maximization Problem

Banks are subject to

$$D_t + A_t = M_t + P_t b_t, \tag{2.10}$$

$$D_t \leq \ell(M_t + \rho P_t b_t). \tag{2.11}$$

The first constraint, Equation (2.10), is the bank's balance sheet identity. The second constraint, Equation (2.11), similar to that in Piazzesi et al. (2019), is a leverage constraint. We impose that $\ell < 1$ and $\rho < 1$. The latter means that other assets have a lower quality compared with

[9] It's easy to allow for market power, but we abstract from it to focus on the main contribution of the paper regarding monetary policy transmission.

[10] Public and private bonds are similar as there is no possibility of default on either of them.

reserves in their use as collateral. We assume that banks do not hold equity. This is without loss of generality because there is no risk of default in the model, so illiqid bonds and equity are essentially the same objects.

Nominal bank profit at time $t+1$ is denoted by Ψ_{t+1} and given by

$$\Psi_{t+1} = P_t b_t \left(1 + i^K_{t+1}\right) + M_t \left(1 + i^M_t\right) - \left(1 + i^D_t\right) D_t - \left(1 + i^S_t\right) A_t.$$

Banks maximize the expected value of discounted profits, which is for time t:

$$\mathscr{R}_t \quad = \quad \boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1}\Psi_{t+1}\right\} = \boldsymbol{E}_t \left\{ \bar{\Lambda}_{t,t+1} \left[\begin{array}{c} P_t b_t \left(1 + i^K_{t+1}\right) + M_t \left(1 + i^M_t\right) \\ -\left(1 + i^D_t\right) D_t - \left(1 + i^S_t\right) A_t \end{array} \right] \right\},$$

where $\bar{\Lambda}_{t,t+1} \equiv \beta U_{C,t+1} / (\pi_{t+1} U_{C,t})$ is the nominal stochastic discount factor.

2.2.2 Bank's Optimality Conditions

We first eliminate A from the bank's problem and then write the Lagrangian for the maximization problem as

$$\boldsymbol{E}_t \left\{ \bar{\Lambda}_{t,t+1} \left[\begin{array}{c} P_t b_t \left(1 + i^K_{t+1}\right) + M_t \left(1 + i^M_t\right) \\ -\left(1 + i^D_t\right) D_t - \left(1 + i^S_t\right) \left(M_t + P_t b_t - D_t\right) \end{array} \right] \right\} + \bar{\lambda}_t \left(-D_t + \ell M_t + \ell \rho P_t b_t\right),$$

where $\bar{\lambda}_t$ is the Lagrangian multiplier associated with the leverage constraint, Equation (2.11).

We then manipulate the optimality conditions to obtain

$$\boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(i^S_t - i^D_t\right)\right\} = \frac{\boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(i^S_t - i^M_t\right)\right\}}{\ell} = \frac{\boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(i^S_t - i^K_{t+1}\right)\right\}}{\ell \rho} = \bar{\lambda}_t,$$

$$\boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^S_t\right)\right\} = 1.$$

Note that the terms inside the expectation above are all known at time t except the interest on the claims on capital and the discount factor Λ_{t+1}, so the conditions can be written as

$$\frac{i^S_t - i^D_t}{1 + i^S_t} = \frac{i^S_t - i^M_t}{\ell \left(1 + i^S_t\right)} = \frac{i^S_t - \frac{\boldsymbol{E}_t \left(\bar{\Lambda}_{t+1} i^K_{t+1}\right)}{\boldsymbol{E}_t \bar{\Lambda}_{t+1}}}{\ell \rho \left(1 + i^S_t\right)} = \bar{\lambda}_t. \tag{2.12}$$

In equilibrium, we must have $i^S_t - i^D_t > 0$ from the money demand, Equation (2.1), so the leverage constraint, Equation (2.11), is always binding, i.e., $\bar{\lambda}_t > 0$. Equation (2.12) is related to the spread between returns on deposits, reserves, and capital. A higher ℓ implies that the assets are better in backing liabilities, so the interest on reserves and the rate of return on capital both decrease. Equation (2.12) is a key equilibrium condition that affects monetary policy transmission in a novel way through banks in our model, as it determines interest rate spreads. For example, we emphasized earlier how the interest rate spread between the bond rate and deposit rate affects Q_t, which in turn appears in optimal labor supply and Euler equations.

Sometimes it is useful to write this equation in terms of the real return on capital, so we will use $1+i^K_{t+1} = \left(1+r^K_{t+1}\right)\pi_{t+1}$, where r^K_{t+1} is the real interest rate on capital and π_{t+1} is the inflation rate at $t+1$.

2.2.3 Monetary Policy Transmission

Before presenting the rest of the model, we discuss at an abstract level how monetary policy transmission works in this model. Monetary policy can work through the interest rate on reserves, the interest rate on CBDC, and the quantity of real balances of reserves and the CBDC.

To elaborate, combine the demand equations for deposits and CBDC along with bank optimality conditions to obtain

$$\underbrace{\frac{i^S_t - i^F_t}{1+i^S_t}}_{\text{CBDC spread}} - \underbrace{\left[\left(1-\frac{\omega_D}{\omega_{FD}}\left(\frac{F_t}{D_t}\right)^{\frac{1}{v}}\right)\underbrace{\frac{i^S_t - i^F_t}{1+i^S_t}}_{\text{CBDC spread}}\right]}_{\text{Spread between CBDC and deposits}}$$

$$= \underbrace{\frac{i^S_t - i^D_t}{\left(1+i^S_t\right)}}_{\text{Deposits spread}} = \frac{1}{\ell}\underbrace{\frac{i^S_t - i^M_t}{\left(1+i^S_t\right)}}_{\text{Reserves spread}} = \frac{1}{\rho\ell}\underbrace{\frac{i^S_t - E_t i^K_{t+1}}{\left(1+i^S_t\right)}}_{\text{Capital return spread}}. \tag{2.13}$$

Equation (2.13) is key for understanding monetary policy transmission in this paper. Here, we highlight how changes in monetary policy rate pass through to other interest rates in the economy. In a typical NK model, there is one interest rate that matters, the one that affects Euler equation, and the central bank directly affects it, and the monetary policy transmits through that to the real economy. In our model, closer to reality, there are several interest rates that agents face and it matters which interest rate, i.e., reserves or CBDC, is set by the central bank, because it affects the spreads differently, which eventually affects households saving decision (through Euler Equation (2.6)) and firm's investment decision (through Equation (2.17)).

The key is that with a non-zero spread, which is the case in our model because of liquidity premium on deposits, interest rates on reserves, deposits and claims to capital can move in the opposite direction relative to the illiquid bond interest rate, regardless of the presence of complementarity between consumption and deposits or the presence of a CBDC. For example, we could have a scenario that following a shock to the quantity of reserves (as in Section 4.2.2), the interest on illiquid bonds declines but the interest rate on reserves, the interest rate on deposits, and (after some initial periods) the interest rate on capital increase. The decrease in the interest rate on illiquid bonds increases consumption through the Euler equation, but the increase in returns of claims to capital pushes the capital stock downwards.

In the general case of Equation (2.13) where CBDC and deposits are not perfect substitutes (and thus, where i^D and i^F are different), effects of i_t^M and i_t^F could be different because the spread (the large term in Equation (2.13)) changes in a complicated way.

In the monetary policy framework based on interest on reserves, i_t^M, a change in the monetary policy interest rate changes the rate on deposits through endogenous response of the banks. The deposit rate change affects the demand side of the economy through money demand, Euler equation, and labor supply conditions. The change in the reserves rate also changes the supply side of the economy again through endogenous response of the banks that changes the rate of return on capital.

In the monetary policy framework based on the interest on CBDC, i_t^F, a change in the monetary policy interest rate changes the demand side of the economy through money demand, Euler equation, and labor supply conditions. This also changes the interest on deposits because CBDC and deposits both compete as means of payments. The change in the deposit rate then changes the rate of return on capital through endogenous response of the banks, which affects the supply side of the economy.

2.3 Production

The production side of the economy is standard similar to NK models with financial constraints (e.g., Gertler and Kiyotaki (2010), Gertler and Karadi (2015), and Gertler and Karadi (2011)). There are three types of non-financial firms: (i) intermediate goods producers, (ii) capital producers, and (iii) monopolistically competitive retailers subject to nominal price rigidity.

2.3.1 Intermediate Goods Producers

Intermediate goods producers use capital and labor according to the following production function:

$$Y_t = F_t K_t^{\alpha} L_t^{1-\alpha},$$

where $\digamma_t$, K_t, and L_t denote productivity, capital and labor. These producers buy new capital from capital-producing firms at the begining of each period and then sell the depreciated capital at the end of the period. The producers do not have funds, so they issue equity (or a perfectly state-contingent debt) to banks and pay back at the end of the period. The price of the claims are denoted by X_t, so

$$X_t K_{t+1} = X_t b_t.$$

The evolution of capital stock is given by:

$$K_{t+1} = I_t + (1-\delta)\xi_t K_t,$$

where ξ_t denotes the capital quality shock and δ is the depreciation rate.

There are no frictions on the side of the intermediate goods producers when financed by banks. The producer's maximization problem is given by

$$\max_{L_t} \left\{ p_{mt} \digamma_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t + \xi_t X_t (1-\delta) K_t \right\},$$

where $p_{mt} = \frac{P_{mt}}{P_t}$ is the real price of the intermediate good. Similarly, $w_t \equiv \frac{W_t}{P_t}$ is the real wage. In a competitive market for labor, the demand for labor is given by

$$w_t = p_{mt}(1-\alpha)\frac{Y_t}{L_t}. \tag{2.14}$$

The profit per unit of capital is given by $Z_t = \alpha p_{mt} \frac{Y_t}{K_t}$.

The maximized value is the revenue of the firm, after paying wages and selling the used capital at the end of the period. Given the initial investment of $X_{t-1}K_t$, the return on investment is given by

$$1 + r_t^K = \frac{\max_{L_t} \left\{ p_{mt} \digamma_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t + X_t \xi_t (1-\delta) K_t \right\}}{X_{t-1} K_t}.$$

The real rate of return on capital is thus summarized as follows:

$$1 + r_t^K = \frac{Z_t + (1-\delta)\xi_t X_t}{X_{t-1}}. \tag{2.15}$$

This denotes the payoff of the firm at time t per unit of capital divided by the price of the claims issued to the banks.

2.3.2 Capital Producers

We follow the model in Gertler and Karadi (2018) for this part. The capital producer turns the final output into capital, denoted by I, subject to adjustment costs, denoted by f function:

$$\max_{I_t} \boldsymbol{E}_t \sum_{\tau=t}^{\infty} \Lambda_{t,\tau} \left\{ X_\tau^i I_\tau - \left[1 + f\left(\frac{I_\tau}{I_{\tau-1}}\right)\right] I_\tau \right\},$$

where $\Lambda_{t,t+i} \equiv \beta^i \frac{U_{c,t+i}}{U_{c,t}}$ denotes the stochastic discount factor. Profit maximization implies that the price of capital goods in terms of the final good must be equal to the marginal cost of investment goods production and is given by the following:

$$X_t = 1 + f\left(\frac{I_t}{I_{t-1}}\right) + \frac{I_t}{I_{t-1}} f'\left(\frac{I_t}{I_{t-1}}\right) - \boldsymbol{E}_t \Lambda_{t,t+1} \left(\frac{I_{t+1}}{I_t}\right)^2 f'\left(\frac{I_{t+1}}{I_t}\right),$$

The profits, which could be non-zero only out of the steady state, are assumed to be distributed between households.

2.3.3 Retailers

Consumption good is the CES composite of a continuum of intermediate goods with a elasticity of substitution ϵ. That is,

$$C_t \equiv \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}.$$

Firm $i \in [0,1]$ produces output from the intermediate good using a linear technology $y_t(i) = h_t(i)$. Firm i sets prices $P_t(i)$ for its good facing demand $y_t(i) = y_t (P_t(i)/P_t)^{-\epsilon}$. The profit at period τ is thus given by $P_\tau(i) h_\tau(i)/P_\tau - d\left(P_\tau(i)/P_{\tau-1}(i)\right) - P_{m\tau} h_\tau(i)$ where $d(.)$ denotes nominal price adjustment cost.

The firm chooses $P_t(i)$ and $h_t(i)$ to maximize expected discounted profits. We substitute out $y_t(i)$ to write the maximization problem as

$$\max \boldsymbol{E}_t \sum_{\tau=t}^{\infty} \Lambda_{t,\tau} \left(\frac{P_\tau(i)}{P_\tau} y_t \left(\frac{P_\tau(i)}{P_\tau}\right)^{-\epsilon} - d\left(\frac{P_\tau(i)}{P_{\tau-1}(i)}\right) - p_{m\tau} y_\tau \left(\frac{P_\tau(i)}{P_\tau}\right)^{-\epsilon} \right).$$

The optimality condition is

$$\begin{aligned} \left[(1-\epsilon)\frac{P_t(i)}{P_t} + \epsilon p_{m\tau}\right] \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} y_t - \frac{P_t(i)}{P_{t-1}(i)} d'\left(\frac{P_t(i)}{P_{t-1}(i)}\right) \\ + \boldsymbol{E}_t \left[\frac{P_{t+1}(i)}{P_t(i)} \Lambda_{t,t+1} d'\left(\frac{P_{t+1}(i)}{P_t(i)}\right)\right] = 0. \end{aligned}$$

Assuming $d(x) = \frac{\kappa}{2}(x-1)^2$, we have $d'(x) = \kappa(x-1)$. Imposing symmetry across firms' price responses (i.e., $P_t = P_t(i)$), we obtain

$$\left[\frac{\epsilon-1}{\epsilon} - p_{m\tau}\right]\frac{\epsilon Y_t}{\kappa} + \frac{P_t}{P_{t-1}}\left(\frac{P_t}{P_{t-1}} - 1\right) = \boldsymbol{E}_t\left[\Lambda_{t,t+1}\frac{P_{t+1}}{P_t}\left(\frac{P_{t+1}}{P_t} - 1\right)\right]. \tag{2.16}$$

Equation (2.16) is a standard Philips curve.

We assume zero capital adjustment costs for the benchmark model (not for calibration), which implies $X_t = 1$. We also assume no capital quality shock throughout the paper, that is, $\xi_t = 1$ for all t, such that Equation (2.15) gives

$$1 + r_t^K = \alpha p_{mt}\frac{Y_t}{K_t} + 1 - \delta, \tag{2.17}$$

or equivalently, $p_{mt} = \frac{r_t^K + \delta}{\alpha}\frac{K_t}{Y_t}$.

2.4 Market Clearing Conditions

The market clearing condition for the final good is given by

$$Y_t = C_t + \left(1 + f\left(\frac{I_t}{I_{t-1}}\right)\right)I_t + \frac{\kappa}{2}\left(\frac{P_t}{P_{t-1}} - 1\right)^2. \tag{2.18}$$

Labor supply and demand, Equations (2.4) and (2.14), together imply

$$Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi L_t^{\varphi} = p_{mt}(1-\alpha)\frac{Y_t}{L_t}.$$

where we used the labor market clearing condition, $H_t = L_t$.

2.5 Monetary and Fiscal Policy

Here, we describe the government's and the central bank's budget constraints as well as the equilibrium condition for the illiquid bond issued by the government and banks.

The government's budget constraint is given by

$$T_t + S_t^G + TR_t = (1 + i_{t-1}^S)S_{t-1}^G,$$

where TR_t denotes the nominal transfer from the central bank to the treasury, T_t denotes the net lump sum tax levied on households, and S_t^G denotes the supply of safe illiquid bonds by the government. The central bank's budget constraint is given by

$$F_t + M_t = TR_t + (1 + i^F_{t-1})F_{t-1} + (1 + i^M_{t-1})M_{t-1}.$$

The illiquid bonds are demanded by households and supplied by banks and the government. The market clearing condition for these bonds is thus given by

$$S^G_t + A_t = S_t.$$

We don't impose any constraint on, T_t, TR_t, and S^G_t, so the equations in this subsection can be solved in a separate block from the rest of equations of the model.[11] That is, T_t, TR_t, and S^G_t are adjusted as needed by these three equations after the determination of equilibrium variables including S_t and A_t. We abstract from matters related to fiscal policy and the relationship between monetary and fiscal authority.

2.6 Steady-State Analysis

In this subsection, we characterize the (non-stochastic) steady-state equilibrium with a zero inflation rate. In the next section, we analyze the effects of various shocks to this economy starting from the zero-inflation rate steady state.

We assume that the central bank sets the interest rates on the CBDC and reserves, i^M and i^F. We drop the subscript t to show the steady-state levels. An alternative policy could be that the central bank sets the real quantity of reserves, for example. We set the aggregate price level to 1. Note that the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. In Appendix A.2, we derive the steady-state equilibrium, summarized in the lemma below.

Lemma 1 *Output in the steady state is given by this closed form solution:*

$$Y^{\varphi+\frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}}(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}+1}}{\psi Q^{1-\frac{\eta}{\sigma}}\left(1-\frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}}\left(i^K+\delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{2.19}$$

This equation gives Y as a function of Q and i^K. Given Y, one can solve for L, C and b.

First, consider the case of the *separable utility* function, $\eta = \sigma$. Output is only a function of i^K,

[11] See Bhattarai et al. (2022) for a discussion of constraints in transfers between the monetary and fiscal authorities. As they show formally in a model, such constraints lead to a role for quantitative easing.

which is pinned down by i^M, given that the bank's leverage constraint is binding. In this case, the interest rate on the CBDC changes only the velocity of money but does not affect output, consumption or investment. The intuition is simple. With a separable utility function, there is no complementarity between money balances and consumption, so the typical channel that exists in cash-in-advance constraint models or NM models would be absent. In this case, an additional unit of money does not help in terms of actual consumption, so the only channel for transmission of monetary policy is through the change in the opportunity cost of lending for banks. As the central bank increases i^M, the opportunity cost of lending rises, so banks tend to hold more reserves and lend less.

Lemma 2 below states conditions under which investment, capital, output and consumption decrease.

Lemma 2 *Assume $\eta = \sigma$. Then,*

(i) Real variables depend on policy only through i^K. So, if i^M is kept fixed, a change in elasticities, ω's, or the CBDC rate does not change real variables.

(ii) An increase in i^M decreases investment, capital , and output; also consumption decreases if δ is sufficiently close to zero.

We provide intuition why the comparative statics with respect to consumption in Lemma 2 is not straightforward and depends on δ. As i^M increases, i^K must increase too (from bank's optimality conditions), so capital becomes more costly. Consequently, the aggregate supply decreases (i.e., Y falls). However, there is a second channel. Since investment in the steady state is a fraction of aggregate capital that needs to be replaced (proportional to the depreciation rate), a decrease in capital decreases demand for investment (i.e., lower I). Since output and investment both decrease, consumption, which is simply the difference between the two, could increase or decrease. If depreciation is small, it means that in the steady state there is less need to replace capital, so investment is small. Therefore, the effect of i^K on output is dominant relative to the effect on investment. That is, with a small depreciation rate an increase in i^K decreases output and consequently consumption.

Next, consider the case of a *non-separable utility* function. For concreteness, assume consumption and money balances are complements. In this case, Y is not only a function of i^K (which is a function of the interest rate on reserves, i^M, as mentioned above) but is also a function of Q, which in turn is a function of the CBDC interest rate. As the interest rate on the CBDC rises, the opportunity cost of holding CBDC balances falls, so the "inflation" tax imposed on consumption declines and consumption increases. Moreover, leisure becomes less valuable, so the supply of labor increases and more output is produced. This result is summarized in Lemma 3 below. We also cover some other special cases in Appendix A.[12]

Lemma 3 *Assume $\eta < \sigma$. Then, an increase in i^F increases output, capital and investment; also consumption increases.*

Using a Lagos-Wright framework, Chiu and Davoodalhosseini (2023) also find that a higher interest rate of a (cash-like) CBDC can improve intermediation, because it increases aggregate demand, leading firms to demand more loans to finance production. A higher CBDC interest rate also makes payments less costly, stimulating consumption. As a result, intermediation, consumption, and output all increase in their model. The mechanism is similar in our model. Our result, however, is more general in that we allow for a rich set of CBDC design features (different degrees of elasticity of substitution). Moreover, we have capital accumulation. Finally, we study the effects of shocks thanks to our framework while they only study the steady state of the model. This is the topic of the next section.

3 Responses to Shocks

In this section and the next, we study the effects of various shocks to this economy by log-linearizing the model around the steady state we characterized in the previous section. We collect all the equilibrium conditions in a model without a CBDC in Appendix A.2, for comparison, as well as equilibrium conditions of the main model with a CBDC in Appendix A.3. We do not repeat all of the equilibrium conditions here except for a few of them that provide insights new to the literature. In terms of notation, the log-linearized version of a variable X_t is denoted by $\widehat{x}_t$. Moreover, $\widetilde{x}_t \equiv \widehat{x}_t - \widehat{p}_t$ for $x \in \{d, f, m\}$ denotes the real balances for deposits, the CBDC, and reserves. Parameters α_{FD}, β_J, α_m, α_c, α_y, and α_{DD} are all constant and defined in Appendix A.3. Finally, we define $\widehat{\pi}_t \equiv \Delta \hat{p}_t$.

[12]There is an *indirect* effect of the CBDC interest rate that operates out of the steady state. An increase in the CBDC interest rate puts pressure on banks to increase the interest on deposits to the extent that CBDC and bank deposits are substitutes. This increases the cost of funding for banks, pushing up the cost of loans for firms. Therefore, the supply side of the economy is negatively affected. This channel does not operate in the steady state because the nominal interest rate is fixed, so the interest rates on deposits and loans are solely determined by the interest on reserves (as long as the leverage constraint is binding) and there is no transmission from the funding to lending sides of the banks. We discuss the effects of CBDC for dynamic responses to shocks in detail in Sections 3 and 4.

3.1 Solving the Model Out of the Steady State

The 18 endogenous variables are as follows: (i) Output, consumption and labor: $\widehat{y}_t, \widehat{c}_t, \widehat{l}_t$; (ii) Output price inflation, real intermediate price, real wage, and the price index for the bundle of consumption and liquidity: $\widehat{\pi}_t$, $\widehat{p}_{mt}$, $\widehat{w}_t$, $\widehat{q}_t$; (iii) Real balances: $\widetilde{d}_t, \widetilde{m}_t, \widetilde{f}_t$; (iv) Capital and investment: $\widehat{k}_t$ and $\widehat{i}_t$; (v) Real bank loans: $\widehat{b}_t$; and finally (vi) Interest rates: i_t^S, r_t^K, i_t^D, i_t^F, i_t^M. There are 15 equations describing the equilibrium optimality and market clearing conditions of the model. See the log-linearized equilibrium conditions in Equation (5.20) to Equation (5.34) in Appendix A. Given that there are 18 unknowns and 15 equations, 3 policy equations are needed to fully close the model. These policy rules should describe the policy variables regarding the interest rate or quantity of various types of liabilities issued by the central bank. The choice of these three variables is at the central bank's discretion, which is described below for different exercises.

In the next section, we focus on reporting the impulse responses to shocks, comparing different model variants as well as policy rules.

3.1.1 Traditional Monetary Policy Rule without a CBDC in the Model

In this benchmark exercise, we assume that a CBDC does not exist in the model. Given that two unknowns (rate and quantity of CBDC) and one equilibrium condition (CBDC demand) is removed, the central bank needs to set only 2 policy rules. We set a rule for interest on reserves and fix the (real) quantity of reserves:

$$i_t^M = r^M + \phi_\pi^M \Delta \hat{p}_t + \phi_y^M \hat{y}_t + u_t^M, \tag{3.1}$$

$$\widetilde{m}_t = u_t^m. \tag{3.2}$$

The terms u_t^M and u_t^m in Equation (3.1) and (3.2) are shocks to the reserves interest rate and reserves quantity, respectively.

3.1.2 Introducing a CBDC into the Model

The next few exercises all include a CBDC in the model.

Exercise A1: Traditional Monetary Policy Rule. In the model with a CBDC, we need to set 3 policy rules. We set a Taylor rule for reserves interest rate and fix the quantity of reserves and CBDC. Therefore, we use the same equations as in the benchmark exercise, Equation (3.1) and

Equation (3.2), as well as the following quantity rule for CBDC:

$$\widetilde{f}_t \quad = \quad u_t^f, \tag{3.3}$$

where the term u_t^f denotes the shock to the CBDC quantity.

Exercise A2: CBDC Rule Instead of Reserves Rule. Monetary policy in the economy above works through the interest rate on reserves. In principle, we can investigate many other rules that the central bank can follow, like rules on the quantity of CBDC or reserves. In particular, we want to compare the implications of the benchmark economy with an economy in which CBDC is the main tool for MP. Therefore, we replace Equation (3.1) with

$$i_t^F = r^F + \phi_\pi^F \Delta \hat{p}_t + \phi_y^F \hat{y}_t + u_t^F. \tag{3.4}$$

The other two equations are the same as those in A1, i.e., Equation (3.2) and (3.3).

Exercise A3: Fixed-Interest CBDC. Many central banks consider only a zero-interest rate CDBC and do not plan to use the interest rate of a CBDC as an active monetary policy tool. Here, we use the same monetary policy rule as in A1, i.e., Equation (3.1), where we assume that the interest rate on the CBDC is zero in the steady state but changes according to an exogenous process:

$$i_t^F = u_t^F, \tag{3.5}$$

where u_t^F is the shock to the interest rate on the CBDC. We now only need one equation to close the model. We use the same reserves quantity rule as in A1, Equation (3.2).

3.2 Analytical Results

The model is fairly complicated and it is hard to derive analytical results in the general case. In this section, we, therefore, focus on some special cases to obtain some analytical results about the responses of the model to interest rate and quantity shocks, and to clarify the main mechanisms at work in our model. In the first part, we focus on the case of no CBDC; in the second part, we focus on the case with CBDC.

3.2.1 Analytical Results with no CBDC

Let's focus on the case without a CBDC. We make these simplifying assumptions in this section: we assume that prices are completely sticky, $\pi_t = 0$, and the depreciation is full, that is, $\delta = 1$, which implies that

$$\widehat{k}_{t+1} = \widehat{b}_t = \widehat{i}_t.$$

In the Appendix, we collect a summary of the log-linearized version of equilibrium conditions with six unknowns: i_t^S, i_t^D, $\widehat{c}_t$, $\widehat{i}_t$, $\widehat{q}_t$, i_t^M. For simplicity, we consider a modified Taylor rule which is a function of consumption instead of output:

$$i_t^M = r^M + \phi_y^M \hat{c}_t + u_t^M. \tag{3.6}$$

We continue by considering two shocks: an unexpected one-time shock to quantity of reserves and an unexpected one-time shock to interest rate on reserves.

Quantity of reserves shock

Now we assume an unexpected one time shock hits the quantity of reserves:

$$\widetilde{m}_t = \Delta \text{ and } \widetilde{m}_t = 0 \text{ for } s > t.$$

Since this is an unexpected shock and dies in just one period, we can set all expected values to zero, in which case we can solve for $i_t^S, \widehat{c}_t, \widehat{i}_t$, $\widehat{q}_t$, i_t^D, i_t^M. In a special case with $\sigma = \eta$, we obtain the following result.

Proposition 1 *With full depreciation, full price stickiness, and no complimentarity between money and consumption, we have the following responses to a one-time shock to the quantity of reserves:*

i. Investment falls iff monetary policy responds sufficiently aggressively to consumption according to the condition below:

$$\phi_y^M > \frac{1-\rho}{\eta\beta\rho} \Leftrightarrow \frac{\partial \widehat{i}_t}{\partial \Delta} < 0. \tag{3.7}$$

ii. Under the same condition, consumption rises:

$$\phi_y^M > \frac{1-\rho}{\sigma\beta\rho} \Rightarrow \frac{\partial \widehat{c}_t}{\partial \Delta} > 0, \tag{3.8}$$

and the interest rate responses are given by:

$$\phi_y^M > \frac{1-\rho}{\sigma\beta\rho} \Rightarrow \frac{\partial \widehat{i}_t^M}{\partial \Delta} > 0, \frac{\partial \widehat{i}_t^D}{\partial \Delta} > 0, \frac{\partial \widehat{i}_t^S}{\partial \Delta} < 0. \tag{3.9}$$

This condition, $\phi_y^M > \frac{1-\rho}{\sigma\beta\rho}$, deserves careful examination. It states that, for a given positive ϕ_y^M, a negative quantity shock to reserves will lead to an increase in capital investment in the economy if and only if ρ is sufficiently close to 1. What is the economic intuition for this result? When ρ is close to 1, claims to capital are treated as near-perfect substitutes for reserves. In the event of a negative shock to the quantity of reserves, a bank can issue claims to capital in amounts nearly equivalent to the reduction in reserves, thereby satisfying its leverage constraint without significantly curtailing deposit issuance. Consequently, investment in the economy must increase to offset the decline in reserves on the bank's balance sheet.

In contrast, when ρ is close to 0, claims to capital are poor substitutes for reserves in meeting the leverage constraint. In this scenario, a bank cannot rely on capital issuance to compensate for the reduction in reserves. As a result, the leverage constraint tightens more substantially, leading to a contraction in deposit issuance. This mechanism operates analogously to a standard negative money quantity shock, exerting contractionary effects on the demand side of the economy and ultimately reducing aggregate capital investment.

Interest rate on reserves shock

Now we assume that an unexpected one-time shock hits the interest on reserves at time t

$$u_t^M = \Delta \text{ and } u_s^M = 0 \text{ for } s > t,$$

and also we assume $\widetilde{m}_t = 0$. Again, the economy is at the steady-state equilibrium and this is an unexpected shock, so we can set all expected values to zero. We can again obtain a linear system $AX = B + \Delta C$, where $X = \left[\widehat{c}_t \; \widehat{i}_t \; i_t^D\right]'$ for some A, B, and C matrices that are functions of the model parameters and defined in the Appendix. Again, we continue to focus on the no-complementarity case, $\eta = \sigma$, for simplicity. Then we have the following proposition.

Proposition 2 *With full depreciation, full price stickiness, and no complimentarity between money and consumption, and assuming that $det(A) < 0$ (defined in the proof), we have the following in response to a one-time shock to interest rate of reserves:*

$$\frac{\rho}{\left(1-\rho\ell\right)\left(i^S - i^D\right)} < \frac{\sigma\beta}{(1-\ell)} + \frac{\varphi\alpha+1}{(1-\alpha)\ell} \Rightarrow \frac{\partial\widehat{i}_t}{\partial\Delta} < 0. \tag{3.10}$$

This result establishes sufficient conditions under which investment declines in response to a one-time increase in the reserves interest rate.

3.2.2 Shock to the CBDC Interest Rate When CBDC and Deposits Are Perfect Substitutes

We now consder the model with CBDC. We are interested in a case where CBDC and deposits are perfect substitutes, but where they are both used in equilibrium. In particular, we assume that $\nu = \upsilon = \infty$, $\omega_{FD} = \omega_D$. That is, CBDC can be used in exactly the same set of transactions as deposits with the same importance. In this case, the deposit and the CBDC interest rates must be the same. We will assess the effects of an exogenous shock to the CBDC interest rate. This type of CBDC is similar to a deposit-like CBDC which has been discussed in the literature, such as in Chiu and Davoodalhosseini (2023) or Keister and Sanches (2023).

Again, as in Section 3.2.1, we assume that prices are completely sticky, the depreciation is full, and the Taylor rule is given by Equation (3.6). After collecting the log-linearized conditions and some algebra, we have the following proposition, with the proof laid out in Appendix C:

Proposition 3 *Assume CBDC and deposits are perfect substitutes. With full depreciation and full price stickiness, we have the following in response to a one-time shock to the CBDC interest rate:*

i. Consumption increases.

ii. Investment falls (i.e., disinvestment) iff monetary policy responds sufficiently aggressively to consumption:

$$\phi_y^M > \frac{1-\rho}{\sigma\beta\rho + \frac{1}{\ell}\frac{\alpha_{DD}}{i^S - i^D}\left(\sigma - \eta\right)} \Leftrightarrow \frac{\partial \widehat{i}_t}{\partial \Delta} < 0. \tag{3.11}$$

iii. There is a range for ϕ_y^M, i.e.,

$$\phi_y^M \in \left(\frac{1-\rho}{\sigma\beta\rho + \frac{1}{\ell}\frac{\alpha_{DD}}{i^S - i^D}\left(\sigma - \eta\right)}, \frac{1-\rho}{\sigma\beta\rho}\right),$$

over which there is disintermediation with complementarity between money and consumption but no disintermediation with neutral money.

This proposition has important messages. First, consumption increases in response to a CBDC interest rate shock with or without complementarity. Second, investment decreases when monetary policy is sufficiently aggressive. With such a monetary policy, interest rate on reserves increases enough to make reserves attractive relative to real investment in the economy. Investment in reserves crowds out real investment, the same message we saw in the previous section. Third, complementarity affects results now. If monetary policy response is modest, then there is disinvestment with complementarity, but no disinvestment with no complementarity. In other words, complementarity qualitatively changes the predictions of the model. Fourth, the responses depend on the aggressiveness of monetary policy.

In what follows, we focus on the case without complementarity to provide sharp insights about the results in this proposition.

Special case: $\sigma = \eta$ It turns out that all we need to know to determine the response of consumption are the following three equations with 3 unknowns $\widehat{c}_t, \widehat{i}_t, i_t^M$:

$$\widehat{c}_t = -\sigma\left(\beta\frac{i_t^M - \ell\Delta}{1-\ell} + \beta - 1\right), \tag{3.12}$$

$$\frac{\left(1-\rho\ell\right) i_t^M - \ell\left(1-\rho\right)\Delta}{\left(1-\ell\right)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t, \tag{3.13}$$

$$i_t^M = r^M + \phi_y^M \hat{c}_t. \tag{3.14}$$

Interestingly, the response of consumption is independent of the response of investment.

$$\widehat{c}_t = \frac{\sigma}{1+\frac{\sigma\beta}{1-\ell}\phi_y^M}\left[-\beta\frac{r^M}{1-\ell} + \left(1-\beta\right) + \beta\frac{\ell\Delta}{1-\ell}\right]. \tag{3.15}$$

This equation is key to understanding the mechanism behind the results in Proposition 3. As i_t^F increases, banks should increase the interest rate on deposits, i_t^D, to keep (some of) their deposits. As interest in deposits increases, i_t^M and i_t^S change. Since i_t^M is determined by the Taylor rule, we should know how consumption responds.

By way of contradiction, assume that consumption decreases, in which case the central bank responds by decreasing the interest rate on reserves, i.e., i_t^M falls. From banks' optimality condition, an increase in interest on deposits, i_t^D, and a decrease in interest on reserves, i_t^M, imply that the interest rate on illiquid bonds, i_t^S, should decrease; otherwise, the marginal cost of holding reserves would be higher than the marginal benefit of issuing deposits, which cannot be true in equilibrium. (That is, $\ell\left(i_t^S - i_t^D\right) = \left(i_t^S - i_t^M\right)$ cannot hold given that $\ell < 1$). Given that i_t^S falls, consumption should rise from the Euler equation, which is a contradiction.

In words, as the interest rate on CBDC increases, the banks respond endogenously and increase the interest rate on deposits. As long as banks hold reserves on their balance sheet, the marginal cost of holding reserves should be equal to the marginal benefit of issuing deposits. This fact implies that the interest rate on illiquid bonds should fall, which leads to an increase in consumption from a standard inter-temporal decision of households.

Note that this mechanism, that consumption increases with the interest rate on the CBDC rate, follows from combination of three equations: (i) Euler equation, (ii) banks' optimality condition on the demand side , i.e., $\ell\left(i_t^S - i_t^D\right) = \left(i_t^S - i_t^M\right)$, and (iii) a Taylor rule which responds to consumption. Some points immediately follow. First, this mechanism is independent of the supply side of the economy. It does not matter how much the rate of return on capital changes. It only depends on the endogenous decision of banks in setting the deposits interest rate and the leverage constraint that they face. Second, the financial friction here is key. Because of financial friction, the interest on illiquid bond has to adjust such that banks have an incentive to issue deposits and hold reserves.

Third, this mechanism is independent of the complementarity channel. When consumption and liquidity are complements, consumption increases additionally because the opportunity cost of liquidity goes down, so the agents would like to increase consumption which is complement to liquidity. Fourth, if the Taylor rule responds to output, which includes investment in addition to capital, then the above argument needs to be modified. If the investment response is negative, the above argument continues to hold, and consumption continues to increase. However, if the response of investment is sufficiently positive and the weight of investment in output is high enough, then output may increase instead (even if consumption decreases). In this case, the central bank might actually want to increase the interest rate on reserves to stabilize output. In this case, an increase in i_t^D and i_t^M together could lead to an increase or decrease in i_t^S, which is consistent with a decrease or increase in consumption.

The following corollary summarizes our results in the special case of no complementarity.

Corollary 1 (Special case of Proposition 3) *Assume CBDC and deposits are perfect substitutes. With full depreciation, full price stickiness, and no complimentarity between money and consumption, consumption is increasing in the interest rate on CBDC, but investment is decreasing iff monetary policy responds aggressively to consumption.*

$$\frac{\partial \widehat{c}_t}{\partial \Delta} = \frac{\frac{\sigma\beta}{1-\ell}}{1 + \frac{\sigma\beta}{1-\ell}\phi_y^M}\ell > 0, \tag{3.16}$$

$$\phi_y^M > \frac{1-\rho}{\sigma\beta\rho} \Leftrightarrow \frac{\partial \widehat{i}_t}{\partial \Delta} < 0. \tag{3.17}$$

4 Quantitative Exercise

In this section, we first parameterize the model and then study the impulse responses of the model to different shocks. In particular, we consider shocks to the interest rate on reserves, the quantity of reserves and the interest rate on CBDC.

4.1 Model Parameterization

The model parameterization is given in Table 1.

Table 1: Parameterization of the model

Parameter	Value	Notes	Source and Target
Households			
β	1/1.01	Utility discount factor	4% Aunnual Discount Rate
σ	0.9	Inverse of relative risk aversion	Close to 1
φ	1	Frisch elasticity of labor supply	Piazzesi et al. (2019).
ψ	1	Disutility of labor	Convention
η	0.6	Substitution elasticity: consumption and deposit	0.22 in Piazzesi et al. (2019).
υ	0.8	Substitution elasticity: deposit and CBDC	New parameter
ω_D	0.5	Utility weight of deposit	0.14 in Piazzesi et al. (2019).
ω_{FD}	0.5	Utility weight of deposit - CBDC	New parameter
Banks			
ℓ	1/1.1	Financial constraint	10% reserve requirement ratio
ρ	0.8	Leverage parameter	80% leverage for non-cash asset
Production			
α	0.35	Technology: capital share	35% Capital usage
δ	0.025/4	Capital depreciation rate	2.5% Annual capital discount factor
κ_I	3	Investment adjustment cost parameter	Justiniano et al. (2011)
ϵ	6	Intermediate good demand elasticity	0.83 Price mark-up
κ_P	75	Price adjustment cost parameter	NK model convention
$\bar{A}$	1	Steady-state TFP	Normalization
Monetary Policy			
ϕ_π	2	Taylor rule, response to inflation	NK model convention
ϕ_Y	0	Taylor rule, response to output	NK model convention
μ_{iM}	0.9	Persistence: Reserves interest rate	NK model convention
μ_{iF}	0.9	Persistence: CBDC interest rate	Match traditional monetary shocks
μ_M	0.9	Persistence: Reserve supply	NK model convention
μ_F	0.9	Persistence: CBDC supply	Match traditional quantity shocks

For most parameters, we use the estimates in the literature. For the zero inflation steady state, we have to set the following parameters. We set the real rate to be around 4%, so the nominal rate should be 4% as well given the zero inflation rate. The nominal rate on reserves is set to $i_M = 3\%$. We also set the interest rate on the CBDC to $i_F = 0$ for the steady state. The values for elasticity of substitution between consumption and means of payments is set as $\sigma = 0.9$, and the intertemporal elasticity of substitution between the consumption bundle today and tomorrow is set as $\eta = 0.6$, both following Piazzesi et al. (2019).

For the substitution between CBDC and deposits, of course, we don't have data. We set it to $\upsilon = 0.8$ or 5 to capture either the case where the CBDC and deposits are complements or where they are substitutes. The model is flexible enough and allows experimentation with different values of design parameters. For example, σ and υ need not be the same. The values for the rest of the parameters are standard in the literature.

4.2 Results in a Model without a CBDC

In this part, we first consider a version of the model without a CBDC and study the reserves interest rate shock and the reserves quantity shock. The policy regime is thus as outlined in Section 3.1.1. The impulse response functions (IRFs) of the endogenous variables to these two shocks are shown in Figure 1 and Figure 2 respectively. In most cases, the directions of responses are as expected. With a positive shock to interest on reserves, output, consumption, and inflation all decrease. With a positive shock to quantity of reserves, all of them increase.

4.2.1 Shock to Interest Rate on Reserves

We start by discussing the results for a shock to the interest rate on reserves, as shown in Figure 1. Since the banks are the only agents in this economy holding reserves, a change in the reserves rate affects only banks and the rest of the effects follow from banks' responses. According to the banks' optimality condition, Equation (2.13), in response to the increase of the reserves interest rate, there is (i) upward pressure on the interest rate of the bond, (ii) downward pressure on the interest rate spread between the bond and the deposit, and (iii) upward pressure on the real return on capital.[13] These three channels interplay, illustrating the mechanisms through which a monetary policy shock influences economic variables.

First, as the nominal bond interest rate i_t^S increases, the traditional NK channel through sticky prices is triggered. An increase in i_t^S leads to adjustments in the current real interest rates in the Euler equation, resulting in reduced current consumption (C_t). This decrease in consumption

[13] In our discussion here when we say that there is an upward (downward) pressure on a certain variable, we mean that an increase (decrease) in that variable is consistent with equilibrium conditions. For some variables, a change in another direction could also be consistent with equilibrium conditions from a theoretical perspective under a counterfactual calibration.

dampens aggregate demand and hence aggregate output (Y_t), subsequently reducing inflation (π_t) through the NK Phillips curve.

Second, the supply channel plays a role whereby the higher marginal cost of capital in the following period ($t+1$) discourages investment, leading to a decline in current output (Y_t). This explains the overall decline in investment along the transition path. There is also a sharp increase in the real rate of capital from the first to second period after the shock. This increase is due to the fact that the capital stock (K_t) is predetermined, with lower output (Y_t) resulting in a decrease in the profit per unit of capital and subsequently real rate of return on capital in the current period t (see Equation (2.15)) but an increase in the subsequent period $t+1$ due to the decrease in current investment.

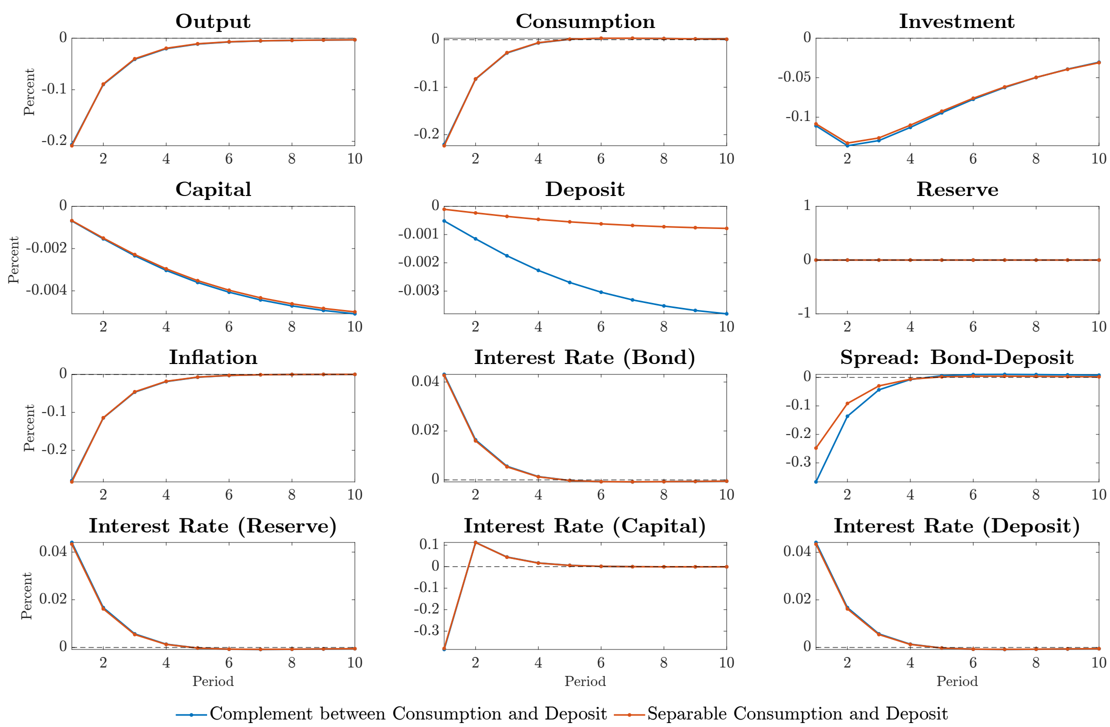

Figure 1: Results without a CBDC: Reserve Interest Rate Shock

Notes: This figure plots the impulse responses to a 1 percent shock to the interest on reserves in a model without CBDC. It compares a model where deposits and consumption are complements with a model where deposits and consumption are separable in utility. See Table 1 for the assigned model parameter values.

Lastly, the NM channel comes into play, wherein the spread between the short-term nominal interest rate (i_t^S) and the deposit rate (i_t^D) affects the cost of liquidity. A narrower spread lowers the cost of liquidity, encouraging higher deposit demand and thus stimulating consumption. This increase in consumption subsequently boosts labor supply and output levels, counteracting the initial dampening effect of the monetary policy shock.

Figure 1 compares a scenario when money is a complement to consumption ($\sigma > \eta$) with one when money is neutral ($\sigma = \eta$), which indicates that the NM channel is shut down. This comparison elucidates the contribution of the NM channel to overall macroeconomic dynamics. The response of output and consumption will slightly increase when the NM channel is shut down, although the effect is quantitatively very small in general equilibrium, except for investment, which falls by less.

4.2.2 Shock to Quantity of Reserves

We now discuss the results for a shock to the quantity of reserves, as shown in Figure 2.

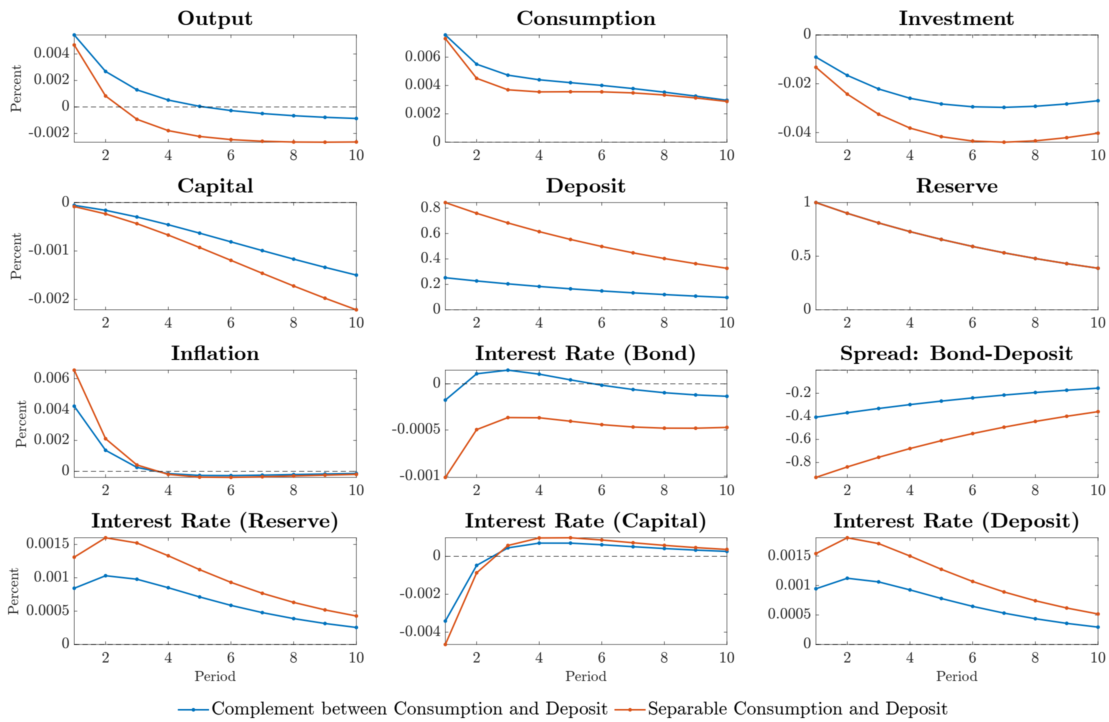

Figure 2: Results without a CBDC: Reserve Quantity Shock

Notes: This figure plots the impulse responses to a 1 percent shock to the quantity of reserves in a model without CBDC. It compares a model where deposits and consumption are complements with a model where deposits and consumption are separable in utility. See Table 1 for the assigned model parameter values.

Compared with a reserve interest rate shock, a reserve quantity shock triggers a different series of adjustments. One key difference is that an increase in reserves supply relaxes the leverage constraint, leading to an expansion of deposits and a reduction in capital accumulation. The reserves expansion leads to several consequential effects even without complementarity between liquidity and consumption (in red). Firstly, reserves expansion reduces the bond-deposit spread, as depicted by the deposit demand equation. This reduction is consistent with lower bond returns, thus stimulating higher consumption, as indicated by the Euler equation. This result is consistent with Proposition 1 for the special case. An increase in deposit rate together with Taylor rule leads to a decrease in the illiquid bond interest rate and increase in consumption. The complementarity between reserves and consumption (in blue) further amplifies this effect, resulting in a larger increase in consumption levels.

Moreover, conventional wisdom might suggest that the reserves interest rate should fall following an increase in the quantity of reserves. However, in this model, the expansion of reserves leads to an expansion of deposits, which requires the opportunity cost of deposits to fall. Banks' optimality conditions Equation (2.12) then imply that the opportunity cost of holding reserves should fall as well. With a lower opportunity cost of holding deposits, consumption and labor supply both increase significantly because of complementarity between consumption and money balances. Given that consumption increases significantly but output does not increase as much, investment should fall. Changes in the investment are supported by a higher rate of return on capital for some periods.

4.3 Comparison of Two Models: No CBDC vs. Zero-Interest CBDC

In this section, we compare the effects of shocks with or without a fixed-interest-rate CBDC, and the interest rate is assumed to be zero unless otherwise noted (such as in Section 4.4). A fixed-interest-rate CBDC represents one of the simplest methods of introducing CBDC and is advocated by many policymakers. The shocks we consider here are a standard monetary policy shock (u_t^{im}) as well as a reserves quantity shock (u_t^{m}). We find that the model with CBDC leads to *larger output and inflation effects* for both types of monetary policy shocks. These results hold for a CBDC that has a low elasticity of substitution with deposits in our model. When CBDC is a perfect substitute (or a high enough substitute) for deposits, we find that the model dynamics are similar to the case without a CBDC in the model.

4.3.1 Shock to Interest Rate on Reserves

The IRFs of a reserves interest rate shock are reported in Figure 3, where we show results for various values of the elasticity of substitution between deposits and CBDC (υ). Comparing the IRFs of the model incorporating CBDC, as here, with the baseline model lacking CBDC (both featuring complementarity between consumption and liquidity), as illustrated in Figure 1, reveals that *the introduction of a CBDC amplifies the impact of a contractionary traditional monetary policy shock*. This amplification arises when CBDC has a low elasticity of substitution with deposits (υ=0.8, shown in red). When the CBDC is highly substitutable with deposits (υ=5, shown in yellow), the model dynamics are similar to the case without a CBDC (shown in blue).

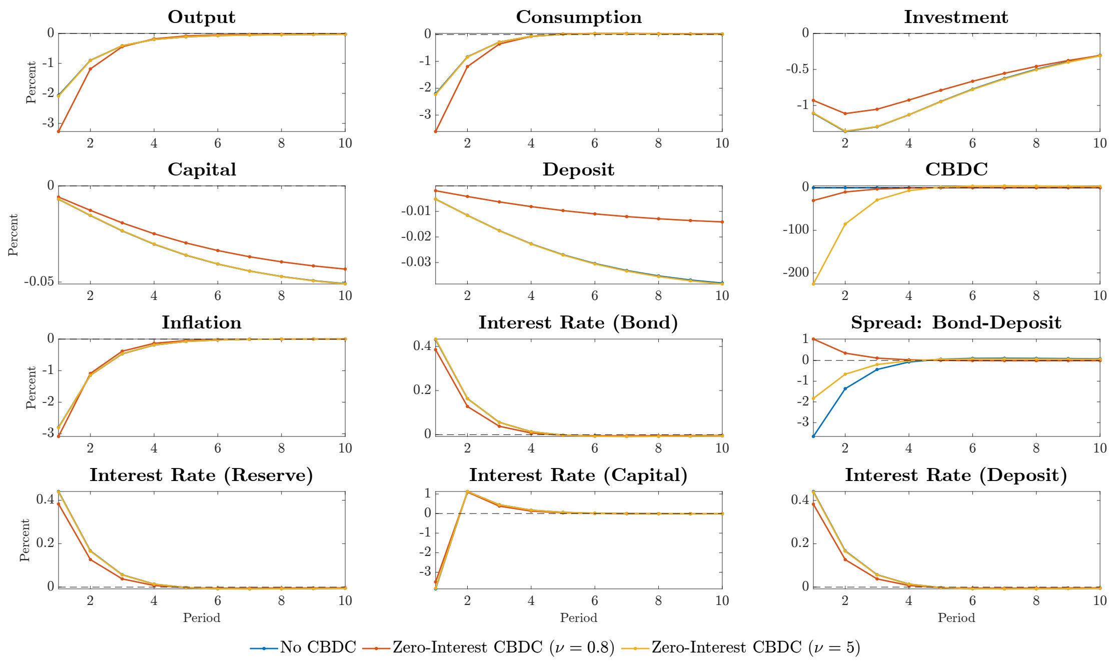

Figure 3: No CBDC vs. Zero-Interest CBDC: Reserves Interest Rate Shock

Notes: This figure plots the impulse responses to a 1 percent shock to the interest on reserves in a model where deposits and consumption are complements. It compares the benchmark model without a CBDC against a model with a zero-interest CBDC. In the zero-interest CBDC case, we consider two alternative values for the substitution elasticity parameter between deposits and CBDC, υ, specifically $\upsilon = 0.8$ (red), and $\upsilon = 5$ (yellow). See Table 1 for the complete list of parameter values.

To better understand why the effects of a traditional monetary policy shock are amplified, we combine the log-linearized version of the demand functions for the CBDC and the deposit, Equations (2.1) and (2.2) with their log-linearized versions shown in Equations (5.21) and (5.22), along with the definition of Q, Equation (2.5), to obtain

$$\widehat{q}_t = \alpha_{DD} \left(\beta_D \left(\frac{i_t^S - i_t^D}{i^S - i^D} - 1 \right) + (1 - \beta_D) \left(\frac{i_t^S - i_t^F}{i^S - i^F} - 1 \right) \right), \tag{4.1}$$

where α_{DD} and β_D are parameters and β_D depends on the steady-state fraction of the deposit in the composition of Q. See derivation in Appendix A.4.

Equation (4.1) is essential to the analysis of an increase in the reserve interest rate shock. In the baseline model in the absence of a CBDC, $\beta_D = 1$ and, hence, the equation will only depend on the spread between i^S and i^D. When the reserves interest rate increases, again using the bank optimality condition Equation (2.12), the bond interest rate tends to increase and the spread between i^S and i^D tends to decrease. In the absence of a CBDC, the decrease in $i^S - i^D$ suggests that Q will decrease. This is the NM channel discussed earlier that can reduce the decrease in total output.

However, with a zero-interest-rate CBDC, i_t^F is zero and, hence, $i^S - i^F$ will increase. Given that β_D always lies between 0 and 1, the introduction of a CBDC diminishes the impact stemming from the decrease in the spread between i^S and i^D. In fact, in equilibrium, the spread increases in the model with CBDC. Consequently, the NM channel is attenuated. Since the NM channel serves to mitigate the decline in output, its attenuation amplifies the decrease in output.

4.3.2 Shock to Quantity of Reserves

The IRFs of a reserves quantity shock are reported in Figure 4 with and without a CBDC (again both featuring complementarity between consumption and liquidity). Note that the latter was already reported in Figure 2. Comparing the effects of the quantity shock across the two scenarios, with and without a CBDC, reveals that the response of consumption, output, and inflation is *stronger* in the case with a CBDC that has a low elasticity of substitution with bank deposits (υ=0.8, shown in red) . When CBDC and bank deposits are highly substitutable (υ=5, shown in yellow), the model dynamics are similar to the case without CBDC (shown in blue).

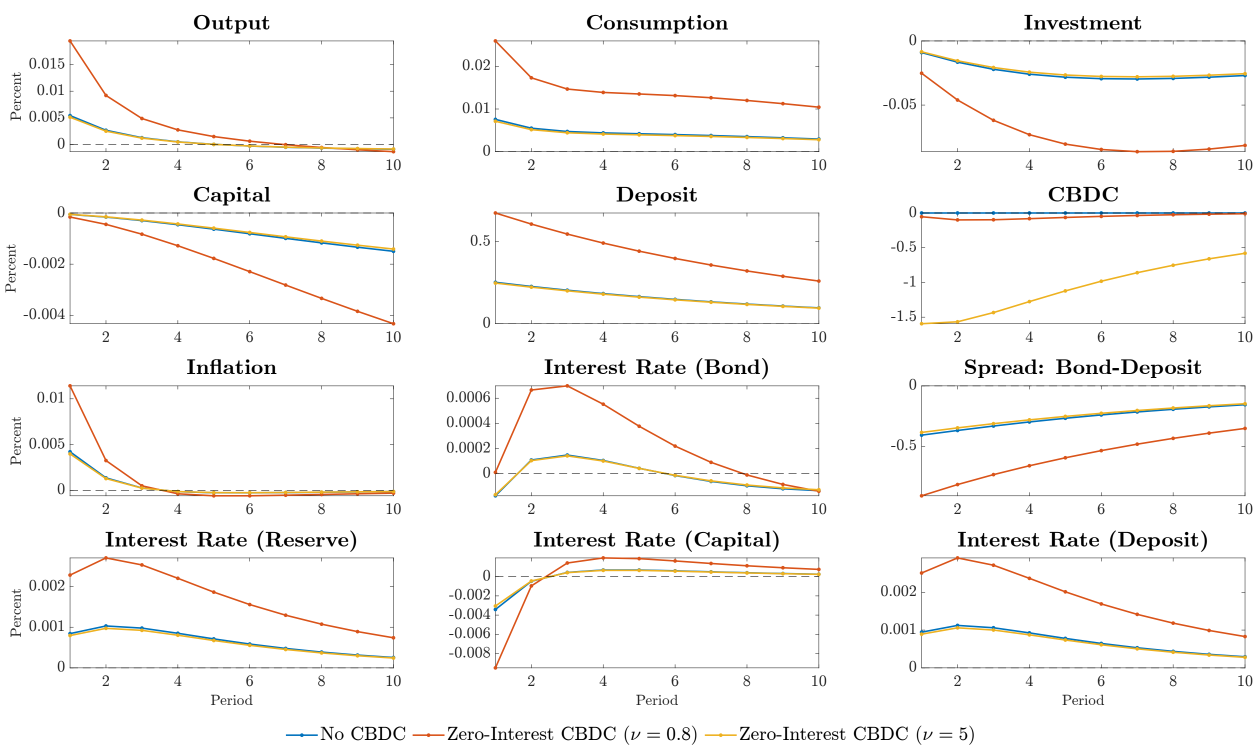

Figure 4: No CBDC vs. a Zero-Interest CBDC: Reserves Quantity Shock

Notes: This figure plots the impulse responses to a 1 percent shock to the quantity of reserves in a model where deposits and consumption are complements. It compares the benchmark model without a CBDC against a model with a zero-interest CBDC. In the zero-interest CBDC case, we consider two alternative values for the substitution elasticity parameter between deposits and CBDC, υ, specifically $\upsilon = 0.8$ (red), and $\upsilon = 5$ (yellow). See Table 1 for the complete list of parameter values.

Here we explain the stronger effects on output, consumption, investment, and inflation in the presence of a CBDC that is an imperfect substitute with deposits. In response to an increase in the quantity of reserves, the spread on reserves must decrease to encourage banks to hold reserves. Lower spread on reserves, and subsequently on deposits, leads to a higher demand for liquidity. Banks respond by offering a higher interest rate on deposits and thus issue more deposits, but the quantity of CBDC demanded declines as the CBDC interest rate cannot respond. The decline in the demand of CBDC is more pronounced when CBDC is a better substitute for deposits. In contrast, when CBDC is a better complement for deposits, it means that CBDC still provides some unique features that households prefer, so the demand for CBDC does not decline as much (compare the yellow and red lines in the CBDC panel in Figure 4). Overall, cheaper deposits lead to a lower price of the bundle of consumption and liquidity (Q), which is more beneficial to consumption when deposits and CBDC are complements because now deposits can take a higher weight in the liquidity bundle.

In the extreme case when CBDC and deposits are perfect substitutes, cheaper deposits do not lead to any amplification because it is as if the CBDC does not exist; so it cannot lead to any change in the composition of liquidity (i.e, β_D is already close to 1). This explains why the consumption response is stronger in the case of an imperfect substitute CBDC. For the same reason, cheaper deposits lead to a higher labor supply (from Equation (2.4)) and potentially higher output (if the capital decline is not too strong). Given that the deposit response is also stronger in the case of an imperfect substitute CBDC and given that quantity of reserves follows an exogenous path, it follows from the leverage constraint that claims to capital and subsequently investment should respond stronger in the case of an imperfect substitute CBDC. The stronger inflation response is explained by the short-term pressure on output, given sticky prices.

4.4 Comparison of Monetary Policy Frameworks: CBDC vs. Reserves

Here, we compare the dynamics of two different policy scenarios. In the first case (Exercise A2), the central bank uses the interest rate on CBDC as the main monetary policy tool, and the quantities of the reserves and CBDC are fixed. In the second case (Exercise A3), the central bank uses the reserves interest rate as the main policy tool, but the interest rate of CBDC follows a simple pre-determined AR(1) process. See Equation (3.5). We compare these two cases in terms of the response to a positive CBDC interest rate shock. The results are reported in Figure 5 which shows that different policy rules can produce in some cases opposite responses.

The main point in this exercise is that the monetary policy tool that is used in response to a CBDC interest rate shock matters significantly. If reserves are used as the main monetary policy tool (shown in red), the CBDC interest rate shock is expansionary. CBDC interest rate payments reduce the opportunity cost of money for households, encouraging higher labor supply and more consumption, the main mechanism that exists in NM models. The NM channel plays an important role in this case, wherein the cost dynamics of money influence economic behavior, ultimately shaping overall economic outcomes. Also, it is interesting to note here that, although output and consumption both expand, investment and consequently capital reduce significantly. Intuitively, CBDC offers an interesting investment opportunity, but this crowds out real investment in this economy.

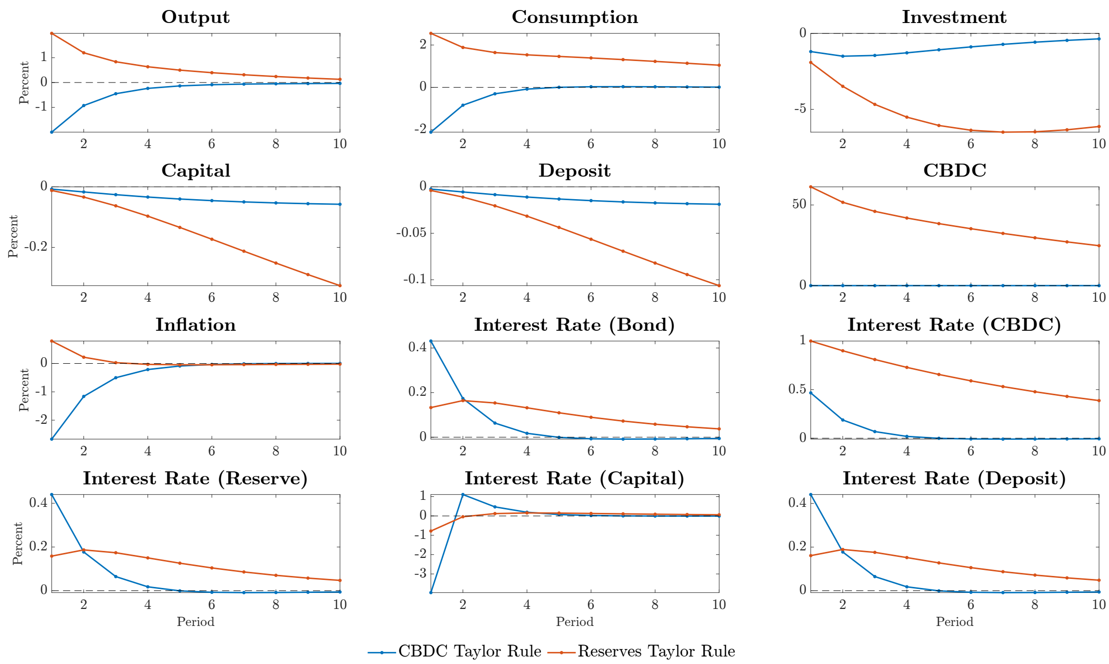

Figure 5: Monetary Policy Tool, CBDC vs. Reserves: CBDC Interest Rate Shock

Notes: This figure plots the impulse responses to a 1 percent shock to the interest on CBDC. It compares a model in which the the interest rate on CBDC is the main monetary policy tool (blue) against a model in which the the interest rate on reserves is the main monetary policy tool (red). See Table 1 for the assigned model parameter values.

When CBDC is utilized as the main tool of monetary policy (shown in blue), an increase in the CBDC interest rate is contractionary, akin to that observed with traditional increase in interest rate on reserves in most NK models. To make that transparent, we plot in Figure 6, the responses to a CBDC interest rate shock when CBDC is the monetary policy tool (shown earlier in Figure 5 above) together with the responses to a reserve interest rate shock when reserves are the monetary policy tool (shown earlier in Figure 3). As is clear, the transmission mechanisms are the same.

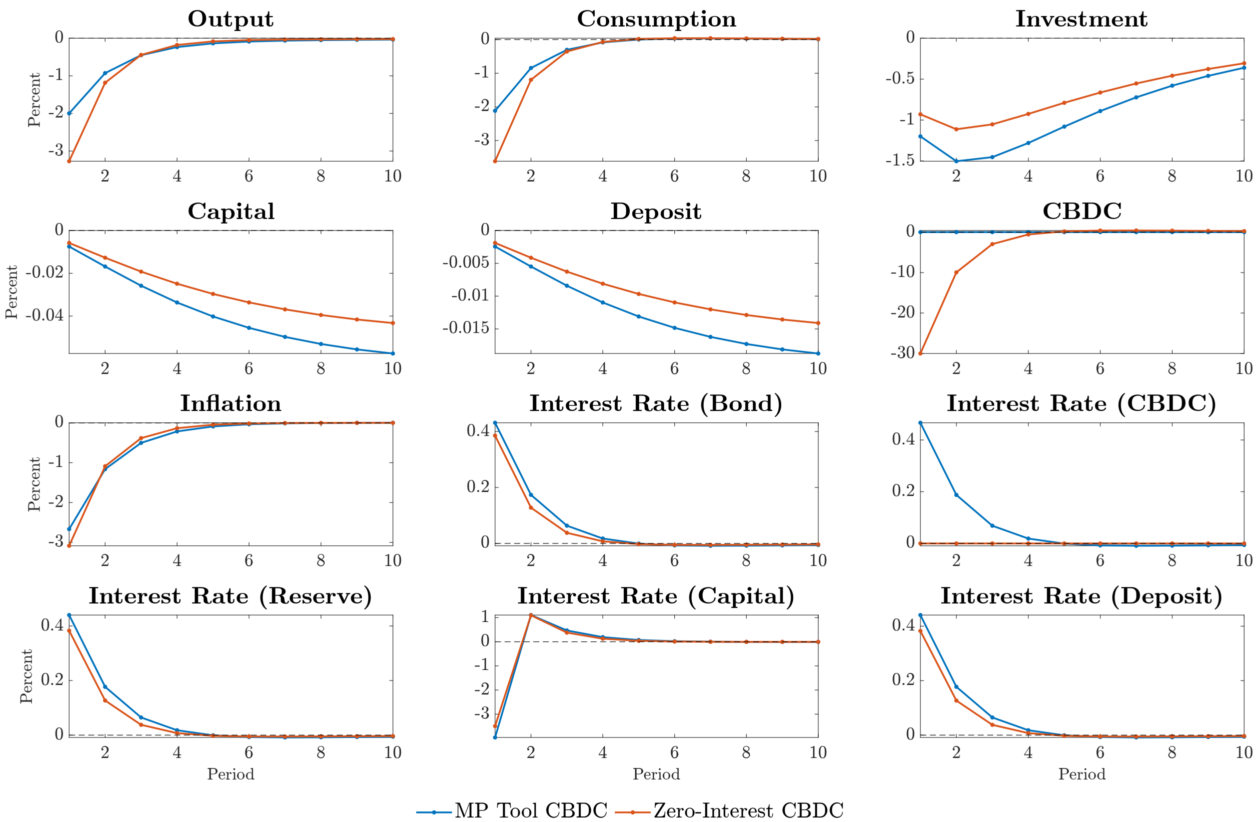

Figure 6: Interest Rate Shock: CBDC vs. Reserves

Notes: This figure plots the impulse responses to a 1 percent shock to the interest on CBDC in a model where the interest rate on CBDC is the main monetary policy tool (blue) and to a 1 percent shock to the interest on Reserves in a model with zero-interest CBDC (red). See Table 1 for the assigned model parameter values.

4.5 Role of Balance Sheet Management

In this section, we demonstrate that the reaction to a standard monetary policy shock, that is a shock to interest rate on reserves, depends on whether the central bank fixes the quantity of CBDC. In other words, we answer the question: is there a role of balance sheet management in standard monetary policy transmission that occurs through changes in interest rate on reserves? To answer this question, we consider two scenarios: one where the central bank fixes the quantity of CBDC and the CBDC interest rate adjusts endogenously (Exercise A1); and another where the central bank fixes the CBDC interest rate and the quantity of CBDC adjusts endogenously (Exercise A3). Figure 7 reports the results.

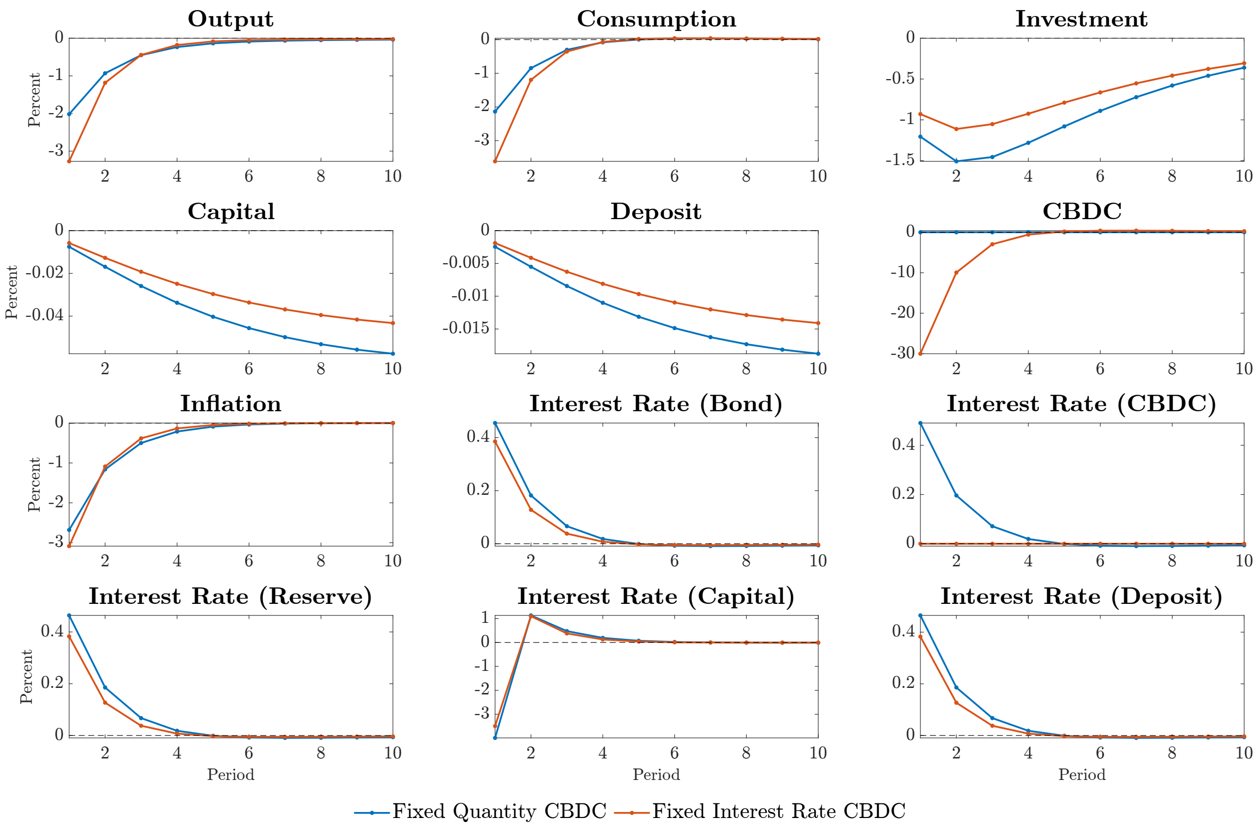

Figure 7: Fixed Quantity of CBDC vs. Fixed Interest Rate of CBDC: Reserves Interest Rate Shock

Notes: This figure plots the impulse responses to a 1 percent shock to the interest on reserves. It compares a model in which the quantity of CBDC is fixed (blue) against a model in which the interest rate on CBDC is fixed (red). See Table 1 for the assigned model parameter values.

As is clear, the policy rule that fixes the CBDC interest rate yields a significant decline in output and consumption compared with the case of the central bank fixing the CBDC quantity. A positive shock to the reserves interest rate puts pressure on interest rates on illiquid bonds and deposits in both cases. When the CBDC rate is flexible, the interest rate on CBDC increases as well, given that CBDC and deposits are imperfect substitutes, activating the NM channel. This partially mitigates the contractionary effects on consumption and output. When the interest rate on CDBC is fixed, the opportunity cost of holding CBDC rises following a rise in the illiquid bonds interest rate. Given the higher opportunity cost of CBDC, liquidity becomes effectively more expensive, leading to a lower level of consumption and output. Moreover, the quantity of CBDC adjusts downwards according to the money demand equation for it.

This exercise illustrates that in a world with CBDC, the balance sheet quantity rule with respect to CBDC matters for determining the response of the economy to a standard monetary policy shock. This is in sharp contrast to standard NK models in which quantity of money is irrelevant for real variables.

4.6 Central Bank Liability Expansions: Reserves vs. CBDC

We now compare the macroeconomic implications of unanticipated expansions in the two central bank liabilities in our model, reserves and CBDC, when reserve interest rate policy follows a Taylor rule. In other words, this policy regime is our baseline Exercise A.1. Figure 8 presents the results. As expected, deposit expansion is much higher when the central bank increases the quantity of reserves than when it increases the quantity of CBDC. Other than this clear difference, in terms of macroeconomic implications, these two policy shocks have similar effects.

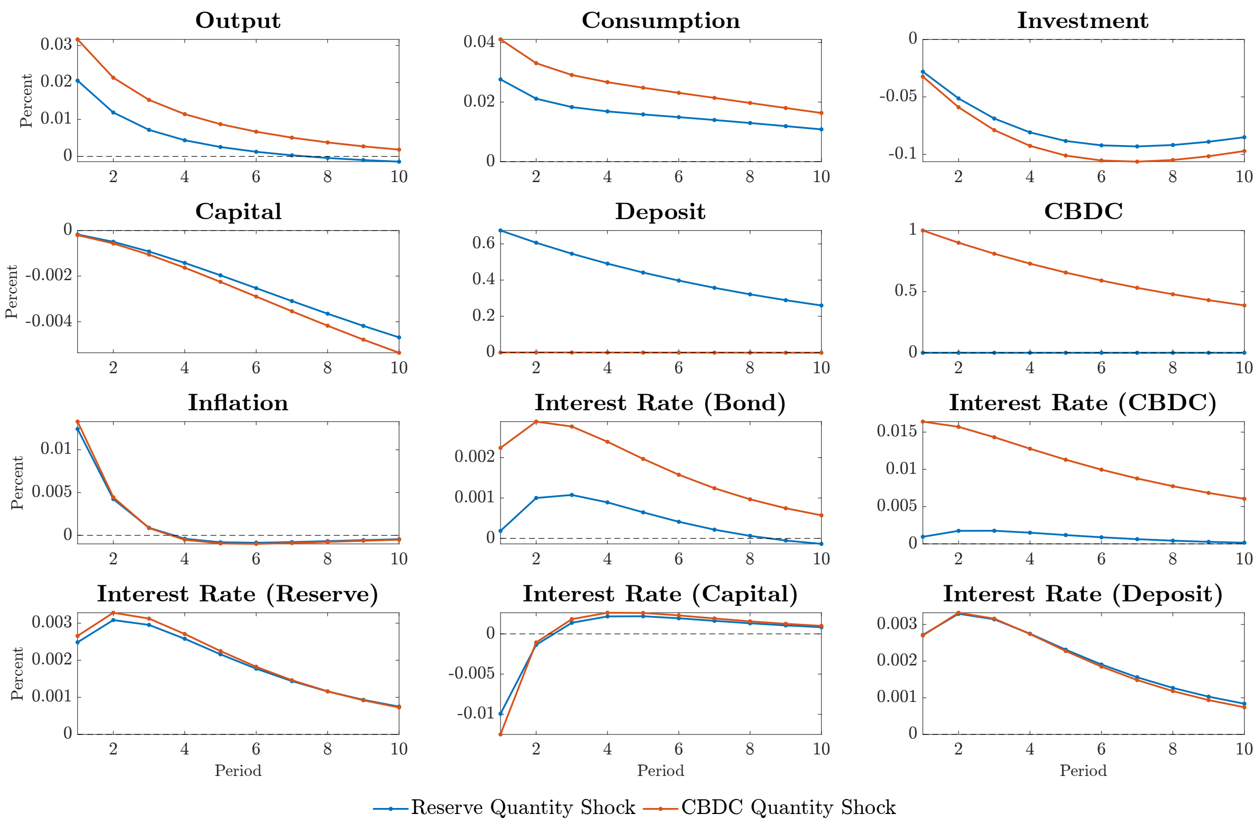

Figure 8: Central Bank Liability Shocks in a Traditional Monetary Policy Rule Regime: Reserve Quantity Shock vs CBDC Quantity Shock

Notes: This figure plots the impulse responses to a 1 percent shock to the quantity of resveres and CBDC. It compares a Reserve quantity shock (blue) against a CBDC quantity shock (red) in a model with a traditional Monetary Policy Rule and quantity rules for CBDC and Reserves. See Table 1 for the assigned model parameter values.

5 Conclusion

Our paper explores the transmission of monetary policy shocks with or without a CBDC using a framework that incorporates nominal rigidities, financial intermediation frictions, and a novel liquidity mechanism. Our analysis reveals that a CBDC that is an imperfect substitute with bank deposits amplifies the effects of monetary policy shocks. Furthermore, our paper underscores the importance of the monetary policy framework adopted by a central bank. Different approaches to setting the interest rate, whether through a CBDC or, as traditionally, through reserves, can lead to markedly different outcomes. When the CBDC interest rate is used as the primary tool of policy, the economy responds to a CBDC interest rate shock in the same manner as it responds to a traditional monetary policy shock in the standard monetary policy framework incorporated in New Keynesian models. However, when reserves are used as the main tool, the response to a positive CBDC interest rate shock can be expansionary, enhancing labor supply and consumption through mechanisms akin to those in New Monetarist models.

Our framework can be used to study other questions in future research. For example, in the appendix we have added cash to the model, which naturally brings about an effective lower bound for nominal rates in this model. One could use this extended version of the model to study the effects of productivity shocks or shocks to banks' financial conditions. One could also introduce quantitative easing in this framework and compare its effects with those of a CBDC, as they look like similar policies in some papers (such as Barrdear and Kumhof (2022)).

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Appendix

A Equilibrium conditions in the benchmark model and in other cases

This appendix consists of several sections.

A.1 Optimality conditions in the benchmark model

Households' optimality conditions can be written as follows:

$$
\begin{aligned}
C: &\quad \frac{U_{C,t}}{P_t} = \lambda_t, \\
J: &\quad \frac{U_{J,t}}{P_t} = \lambda_t - \beta \boldsymbol{E}_t \lambda_{t+1}\left(1+i_t^J\right) \text{ for } J \in \{D, F\}, \\
S: &\quad \lambda_t = \beta \boldsymbol{E}_t \lambda_{t+1}\left(1+i_t^S\right), \\
H: &\quad U_{H,t} = -\lambda_t W_t.
\end{aligned}
$$

A.2 Equilibrium conditions for the benchmark model (without a CBDC and cash)

$$
\text{Illiquid bond demand:} \quad \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}}\right]\left(1+i_t^S\right) = 1
$$

$$
\text{Deposit demand:} \quad \frac{i_t^S - i_t^D}{1+i_t^S} = \omega_D\left(\frac{P_t C_t}{D_t}\right)^{\frac{1}{\eta}}
$$

$$
\text{Bank FOCs:} \quad \frac{i_t^S - i_t^D}{1+i_t^S} = \frac{i_t^S - i_t^M}{\left(1+i_t^S\right)\ell} = \frac{i_t^S - E_t i_{t+1}^K}{\left(1+i_t^S\right)\rho\ell}
$$

$$
\text{Production and market clearing:} \quad Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} = C_t + I_t + \frac{\kappa}{2}\left(\frac{P_t}{P_{t-1}} - 1\right)^2
$$

$$
K_{t+1} = b_t = I_t + (1-\delta) K_t
$$

$$\text{Labor demand:} \qquad w_t = p_{mt}(1-\alpha)\frac{Y_t}{L_t}$$

$$\text{Capital demand:} \qquad 1 + r_t^K = \alpha p_{mt}\frac{Y_t}{K_t} + 1 - \delta$$

$$\text{Labor supply:} \qquad w_t = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi L_t^{\varphi}$$

$$\rightarrow p_{mt} Y_t = \frac{\left(\delta + r_t^K\right) K_t}{\alpha} = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi \frac{L_t^{1+\varphi}}{(1-\alpha)}$$

$$\text{Optimal pricing:} \quad \left[\frac{\epsilon - 1}{\epsilon} - p_{mt}\right]\frac{\epsilon Y_t}{\kappa} + \frac{P_t}{P_{t-1}}\left(\frac{P_t}{P_{t-1}} - 1\right) = E_t\left[\Lambda_{t,t+1}\frac{P_{t+1}}{P_t}\left(\frac{P_{t+1}}{P_t} - 1\right)\right]$$

$$Q_t \equiv \left(1 + \omega_D V_{D,t}^{-(1-\frac{1}{\eta})}\right)^{\frac{1}{1-\eta}}$$

$$\text{Leverage constraint } \frac{M_t}{P_t} = \frac{1}{\ell}\frac{D_t}{P_t} - \rho b_t$$

Unknowns:

- C, Y, K, L, I, Q
- $p_{mt}, \frac{P_{t+1}}{P_t}$
- $b, \frac{D_t}{P_t}, \frac{M_t}{P_t}$
- i_t^S, i_t^D, i_t^K

These are 13 equations and 14 unknowns, so we need an equation to close the model:

$$\text{Market clearing for reserves: } \frac{M_t^S}{P_t} = \frac{M_t}{P_t}.$$

Exogenous: $i_t^M, \frac{M_t^S}{P_t}$. [14] [15]

[14]Note that the leverage constraint must be binding all the time. If the constraint is slack, we must have $i_t^S = i_t^D = i_t^M = E_t i_{t+1}^K$; but in that case, the deposit demand equation implies that D should be very large, which means that the constraint cannot be slack!

[15]If the central bank sets the **nominal** value of reserves, then we have the following unknowns:

Steady State Analysis

Proof of Lemma 1. In the steady state we have:

$$\text{Intermediate good price:} \qquad p_m = \frac{\epsilon-1}{\epsilon},$$
$$\text{Illiquid bond demand:} \qquad \beta\left(1+i^S\right) = 1.$$

Now we continue in three steps.

Block 1: Given i^M, we can solve for i^K and i^D:

$$\frac{i^S - i^D}{1+i^S} = \frac{i^S - i^M}{\left(1+i^S\right)\ell} = \frac{i^S - i^K}{\left(1+i^S\right)\rho\ell}. \tag{5.1}$$

Block 2: Given i^D from block 1, we can pin down $V_D \equiv \frac{PC}{D}$:

$$\text{Deposit demand: } \frac{i^S - i^D}{1+i^S} = \omega_D V_D^{\frac{1}{\eta}}.$$

Moreover, we have:

$$b = K = \frac{I}{\delta}.$$

Block 3: Given i^K (from block 1) and $Q \equiv \left(1+\omega_D V_D^{-(1-\frac{1}{\eta})}\right)^{\frac{1}{1-\eta}}$ (from block 2), the following four equations pin down Y, L, C and b:

$$\begin{aligned}
Y &= C + \delta K, \\
p_m &= \frac{\epsilon-1}{\epsilon} = \frac{\left(i^K+\delta\right)K}{\alpha Y} \rightarrow K = \frac{\epsilon-1}{\epsilon}\frac{\alpha Y}{i^K+\delta}, \\
Y &= AK^{\alpha}L^{1-\alpha}, \\
\frac{\epsilon-1}{\epsilon}Y &= Q^{1-\frac{\eta}{\sigma}}C^{\frac{1}{\sigma}}\psi\frac{L^{1+\varphi}}{1-\alpha}.
\end{aligned} \tag{5.2}$$

We can now calculate Y as a function of C and K, and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon-1}{\epsilon}\frac{(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}K^{\alpha\frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}}C^{\frac{1}{\sigma}}\psi},$$

$C, Y, K, L, I; p_{mt}, P_t; b, D_t, M_t; i_t^S, i_t^D, i_t^K$.The market clearing for reserves remains the same, $M_t^S = M_t$.

$$Y = C + \delta K \rightarrow C = Y\left(1 - \frac{\epsilon - 1}{\epsilon}\frac{\alpha\delta}{i^K + \delta}\right). \tag{5.3}$$

Given that K and C are now given in terms of Y from the last two numbered equations, we obtain

$$Y^{\varphi+\frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}}(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}+1}}{\psi Q^{1-\frac{\eta}{\sigma}}\left(1-\frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}}\left(i^K+\delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{5.4}$$

■

This concludes the proof, but we can use the following step to finish the analysis of the steady state.

Block 4, a binding leverage constraint gives M/P:

$$\frac{D}{P} = \ell\left(\frac{M}{P} + \rho b\right), \tag{5.5}$$

where $b = K$ comes from block 3 and $D/P = CV_D^{-1}$ comes from block 1 (for V_D) and block 3 (for C).

At the end, $w = \frac{\epsilon-1}{\epsilon}(1-\alpha)\frac{Y}{L}$ gives w, and $I = \delta K$. Also, $F/P = CV_F^{-1}$ gives F.

Proof of Lemma 2. (i): obvious from (2.19).
(ii): note that an increase in i^M increases i^K through (5.1), which **decreases output** through (2.19) and capital through (5.2), and consequently I decreases because I=δK. For C, combine (5.3) and (2.19) to obtain

$$\begin{aligned} C &= const.\left(1-\frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right)^{\frac{\sigma\varphi}{1+\sigma\varphi}}\left(i^K+\delta\right)^{\frac{-\alpha(1+\varphi)\sigma}{(1-\alpha)(1+\sigma\varphi)}} \\ &\rightarrow \frac{\partial \ln C}{\partial i^K}\left(\frac{i^K+\delta}{\sigma}\right)\left(\frac{1+\sigma\varphi}{\varphi}\right) = \frac{1}{\frac{(i^K+\delta)}{\frac{\epsilon-1}{\epsilon}\alpha\delta}-1} - \frac{\alpha}{1-\alpha}\frac{1+\varphi}{\varphi} \\ &\rightarrow \frac{\partial \ln C}{\partial i^K} < 0 \Leftrightarrow 1 + \frac{1-\alpha}{\alpha}\frac{\varphi}{1+\varphi} < \frac{i^K+\delta}{\frac{\epsilon-1}{\epsilon}\alpha\delta}, \end{aligned}$$

which is true if δ is sufficiently close to zero. ■

Proof of Lemma 3. An increase in i^F decreases Q according to Equation (2.9), which increases Y from Equation (2.19). ■

Log-Linearized Version:

$$\text{Euler equation: } \widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right] - \sigma\left(\beta i_t^S - E_t\left[\widehat{\pi}_{t+1}\right] + \beta - 1\right) + (\sigma - \eta)\left(E_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \tag{5.6}$$

$$\text{Deposit demand:} \frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{d_t} \tag{5.7}$$

Bank equations:

$$\text{Bank FOC:} \quad i_t^S - i_t^D = \ell^{-1}\left(i_t^S - i_t^M\right) \tag{5.8}$$

$$\text{Bank FOC} \quad : \quad i_t^S - E_t r_{t+1}^K - \left(1 + r^K\right) E_t\left[\widehat{\pi}_{t+1}\right] = \rho\left(i_t^S - i_t^M\right) \tag{5.9}$$

$$\text{Bank Leverage} \quad : \quad \widetilde{d} = \alpha_m \widetilde{m} + (1 - \alpha_m)\,\widehat{b}_t \tag{5.10}$$

Philips curve:

$$\widehat{\pi}_t = \frac{(\epsilon - 1)\,Y}{\kappa}\widehat{p}_{mt} + \beta E_t\left[\widehat{\pi}_{t+1}\right] \tag{5.11}$$

The rest of the equations:

$$\widehat{y}_t = \alpha \widehat{k}_t + (1 - \alpha)\,\widehat{l}_t \tag{5.12}$$

$$\widehat{y}_t = \alpha_c \widehat{c}_t + (1 - \alpha_c)\,\widehat{i}_t \tag{5.13}$$

$$\widehat{k}_{t+1} = \widehat{b}_t \tag{5.14}$$

$$\widehat{k}_{t+1} = \delta \widehat{i}_t + (1 - \delta)\,\widehat{k}_t \tag{5.15}$$

$$\widehat{w}_t = \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t \tag{5.16}$$

$$\frac{r_t^K - r^K}{1 + r^K} = \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) \tag{5.17}$$

$$\widehat{w}_t = \left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi \widehat{l}_t \tag{5.18}$$

$$\widehat{q}_t = \frac{1}{\eta}\alpha_{DD}\widehat{V}_{D,t} = \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \widetilde{d}_t\right),$$

where

$$\alpha_{DD} \equiv \frac{\omega_D V_D^{\frac{1}{\eta}-1}}{1+\omega_D V_D^{\frac{1}{\eta}-1}}.$$

$$\text{Reserves interest rate (Taylor) rule}: i_t^M = r^M + \phi_\pi^M \Delta \hat{p}_t + \phi_y^M \hat{y}_t + u_t^M \tag{5.19}$$

$$\text{Reserves quantity rule}: \widetilde{m} = u_t^{\text{Reserves}}$$

A.3 Problem with a CBDC

Using the following definitions:

$$\begin{aligned}
\widetilde{x} &= \widehat{x}_t - \widehat{p}_t \text{ for } x \in \{d, f, m\}, \\
\alpha_{FD} &\equiv 1, \\
\beta_D &\equiv \frac{V_D^{-(1-\frac{1}{v})}}{V_D^{-(1-\frac{1}{v})} + \frac{\omega_{FD}}{\omega_D} V_F^{-\left(1-\frac{1}{v}\right)}}, \\
\alpha_m &\equiv \frac{M/P}{M/P+\rho b}, \\
\alpha_c &\equiv \frac{C}{Y}, \\
\alpha_y &\equiv \frac{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K}}{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K} + 1 - \delta}, \\
\alpha_{DD} &\equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1+\omega_D V_{FD}^{\frac{1}{\eta}-1}}.
\end{aligned}$$

Here is a summary of the log-linearized version of equilibrium conditions:

$$\text{Euler equation: } \widehat{c}_t = E_t[\widehat{c}_{t+1}] - \sigma\left(\beta i_t^S - E_t[\widehat{\pi}_{t+1}] + \beta - 1\right) + (\sigma - \eta)\left(E_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \tag{5.20}$$

$$\text{Deposit demand: } \frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \left(\frac{1-\beta_D}{v} + \frac{\beta_D}{\eta}\right)\widetilde{d} - \left(-\frac{1}{v} + \frac{1}{\eta}\right)(1-\beta_D)\widetilde{f} \tag{5.21}$$

$$\text{CBDC demand: } \frac{i_t^S - i_t^F}{i^S - i^F} - 1 = \frac{1}{\eta}\widehat{c}_t - \left(\frac{1}{\eta} - \frac{1}{v}\right)\beta_D \widetilde{d} - \left(\frac{1-\beta_D}{\eta} + \frac{\beta_D}{v}\right)\widetilde{f} \tag{5.22}$$

$$\text{Bank FOC:} \quad i_t^S - i_t^D = \ell^{-1}\left(i_t^S - i_t^M\right) \tag{5.23}$$

$$\text{Bank FOC:} \quad i_t^S - E_t r_{t+1}^K - \left(1+r^K\right) E_t\left[\widehat{\pi}_{t+1}\right] = \rho\left(i_t^S - i_t^M\right) \tag{5.24}$$

$$\text{Bank Leverage:} \quad \widetilde{d} = \alpha_m \widetilde{m} + (1-\alpha_m)\,\widehat{b}_t \tag{5.25}$$

$$\text{Philips curve: } \widehat{\pi}_t = \frac{(\epsilon-1)\,Y}{\kappa}\widehat{p}_{mt} + \beta E_t\left[\widehat{\pi}_{t+1}\right] \tag{5.26}$$

The rest of equations:

$$\widehat{y}_t = \alpha\widehat{k}_t + (1-\alpha)\,\widehat{l}_t \tag{5.27}$$

$$\widehat{y}_t = \alpha_c\widehat{c}_t + (1-\alpha_c)\,\widehat{i}_t \tag{5.28}$$

$$\widehat{k}_{t+1} = \widehat{b}_t \tag{5.29}$$

$$\widehat{k}_{t+1} = \delta\widehat{i}_t + (1-\delta)\,\widehat{k}_t \tag{5.30}$$

$$\widehat{w}_t = \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t \tag{5.31}$$

$$\frac{r_t^K - r^K}{1+r^K} = \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) \tag{5.32}$$

$$\widehat{w}_t = \left(1-\frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi\widehat{l}_t \tag{5.33}$$

$$\widehat{q}_t = \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \left(\beta_D\widetilde{d} + (1-\beta_D)\widetilde{f}_t\right)\right) \tag{5.34}$$

Here are the exercises we do.

<u>Exercise A1:</u>

$$i_t^M = r^M + \phi_\pi^M \Delta\hat{p}_t + \phi_y^M \hat{y}_t + u_t^M. \tag{5.35}$$

We need 2 more equations to close the model:

$$\widetilde{f}_t = u_t^f. \tag{5.36}$$

$$\widetilde{m}_t = u_t^m. \tag{5.37}$$

We can shock these variables, i_t^M, $\widetilde{f}_t$, $\widetilde{m}_t$ one by one.

Exercise A2:

$$i_t^F = r^F + \phi_\pi^F \Delta\hat{p}_t + \phi_y^F \hat{y}_t + u_t^F. \tag{5.38}$$

We need 2 more equations to close the model:

$$\widetilde{f}_t = u_t^f. \tag{5.39}$$

$$\widetilde{m}_t = u_t^m. \tag{5.40}$$

We can shock these variables, i_t^F, $\widetilde{f}_t$, $\widetilde{m}_t$ one by one.

Steady State Equations when the CBDC and deposits are perfect substitutes

$\upsilon = \infty$ and $\omega_D = \omega_{FD}$.

Steady State Equations:

- Output, consumption and labor: Y, C, L
- Deposits, CBDC and reserves balances: D, F, M
- Real assets: b
- Rates: i^K, i^D

Note that the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. We now derive the steady state values:

$$\text{Intermediate good price:} \qquad p_m = \frac{\epsilon - 1}{\epsilon},$$

$$\text{Illiquid bond demand:} \qquad \beta\left(1 + i^S\right) = 1.$$

Given the perfect substitution assumption, we have

$$\text{Perfect substitution:} \qquad i^D = i^F.$$

Note that i^M and i^F cannot be two independent policy tools. Rather, one can be calculated from the other. Here, we assume i^M is given.

Block 1: Given i^M, we can solve for i^K and $i^D = i^F$:

$$\frac{i^S - i^D}{1+i^S} = \frac{i^S - i^M}{\left(1+i^S\right)\ell} = \frac{i^S - i^K}{\left(1+i^S\right)\rho\ell}. \tag{5.41}$$

Block 2: Given i^D from block 1 and i^F from policy, we can pin down V_D and V_F:

$$\text{Deposit and CBDC demand:} \qquad \frac{i^S - i^D}{1+i^S} = \omega_D Q_D^{-\frac{1}{\eta}}. \tag{5.42}$$

Given $Q_D = \frac{D+F}{PC}$, we can calculate $Q \equiv \left(1+\omega_D Q_D^{1-\frac{1}{\eta}}\right)^{\frac{1}{1-\eta}}$.

Moreover, we have

$$b = K = \frac{I}{\delta}.$$

Block 3: Given i^K (from block 1) and Q (from block 2), the following four equations pin down Y, L, C and b:

$$\begin{aligned} Y &= C + \delta K, \\ p_m &= \frac{\epsilon-1}{\epsilon} = \frac{\left(i^K+\delta\right)K}{\alpha Y} \rightarrow K = \frac{\epsilon-1}{\epsilon}\frac{\alpha Y}{i^K+\delta}, \\ Y &= AK^\alpha L^{1-\alpha}, \\ \frac{\epsilon-1}{\epsilon}Y &= Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi \frac{L^{1+\varphi}}{1-\alpha}. \end{aligned} \tag{5.43}$$

We can now calculate Y as a function of C and K and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon-1}{\epsilon}\frac{(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}K^{\alpha\frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}}C^{\frac{1}{\sigma}}\psi},$$

$$Y = C + \delta K \rightarrow C = Y\left(1 - \frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right). \tag{5.44}$$

Given that K and C are now given in terms of Y from Equation (5.43) and Equation (5.44), we obtain

$$Y^{\varphi+\frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}}(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}+1}}{\psi Q^{1-\frac{\eta}{\sigma}}\left(1-\frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}}\left(i^K+\delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{5.45}$$

$$
\begin{aligned}
Q_D &\equiv V_D^{-1} + V_F^{-1} = \frac{D+F}{PC} \\
Q &\equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}}.
\end{aligned}
$$

Block 4, a binding leverage constraint gives M/P:

$$
\frac{D}{P} = \ell\left(\frac{M}{P} + \rho b\right), \tag{5.46}
$$

where $b = K$ comes from block 3 and $(D+F)/P = CQ_D$ comes from block 1 (for V_{FD}) and from block 3 (for C).

At the end, $w = \frac{\epsilon-1}{\epsilon}(1-\alpha)\frac{Y}{L}$ gives w, and $I = \delta K$.

The log-linearized version of equilibrium conditions is no different than the model with general υ. We just need to set υ to a sufficiently large number.

A.4 Derivation of Equation (4.1)

First, combine the log-linearized version of the demand functions for the CBDC and the deposit, Equations (2.1) and (2.2) with their log-linearized versions shown in Equations (5.21) and (5.22), to obtain

$$
\widetilde{d}_t - \widetilde{f}_t = \upsilon\left(\frac{i_t^S - i_t^F}{i^S - i^F} - \frac{i_t^S - i_t^D}{i^S - i^D}\right), \tag{5.47}
$$

which suggests that the difference between the demands of different assets depends on the difference of the spreads. Using (5.47), we can rearrange the demand functions for the deposit and CBDC:

$$
\frac{i_t^S - i_t^D}{i^S - i^D} - 1 - (1-\beta_D)\left(\frac{\upsilon}{\eta} - 1\right)\left(\frac{i_t^S - i_t^F}{i^S - i^F} - \frac{i_t^S - i_t^D}{i^S - i^D}\right) = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{d}_t, \tag{5.48}
$$

$$
\frac{i_t^S - i_t^F}{i^S - i^F} - 1 + \beta_D\left(\frac{\upsilon}{\eta} - 1\right)\left(\frac{i_t^S - i_t^F}{i^S - i^F} - \frac{i_t^S - i_t^D}{i^S - i^D}\right) = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{f}_t. \tag{5.49}
$$

We use Equations (5.48) and (5.49) to eliminate the quantity of the deposit and CBDC in the definition of Q, Equation (2.5), to obtain Equation (4.1).

B Proofs of Propositions 1 and 2

Here we focus on the case without a CBDC with extreme assumptions: full depreciation, $\delta = 1$, full price stickiness, $\kappa = \infty$, and exogenous $\widetilde{m}$.

Use the following definitions:

$$\begin{aligned}
\widetilde{x} &= \widehat{x}_t - \widehat{p}_t \text{ for } x \in \{d, f, m\}, \\
\alpha_{FD} &\equiv 1, \\
\beta_D &\equiv 1, \\
\alpha_m &\equiv \frac{M/P}{M/P + \rho b}, \\
\alpha_c &\equiv \frac{C}{Y}, \\
\alpha_y &\equiv \frac{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K}}{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K} + 1 - \delta}, \\
\alpha_{DD} &\equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}}.
\end{aligned}$$

Here is a summary of the log-linearized version of equilibrium conditions:

$$\widehat{\pi}_t = 0$$

$$\text{Euler equation: } \widehat{c}_t = E_t[\widehat{c}_{t+1}] - \sigma\left(\beta i_t^S + \beta - 1\right) + \left(\sigma - \eta\right)\left(\mathbb{E}_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \tag{5.50}$$

$$\text{Deposit demand: } \quad \frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{d_t} \tag{5.51}$$

$$\begin{aligned}
\text{Bank FOC:} \quad & \ell\left(i_t^S - i_t^D\right) = \left(i_t^S - i_t^M\right) \Rightarrow i_t^S = \frac{i_t^M - \ell i_t^D}{1 - \ell} \\
\text{Bank FOC:} \quad & i_t^S - \mathbb{E}_t r_{t+1}^K = \rho \ell\left(i_t^S - i_t^D\right) = \rho \ell \frac{i_t^M - i_t^D}{1 - \ell} \\
\text{Bank Leverage:} \quad & \widetilde{d}_t = \alpha_m \widetilde{m} + (1 - \alpha_m)\,\widehat{i}_t
\end{aligned} \tag{5.52}$$

So,

$$i_t^S - i_t^M = \ell \frac{i_t^M - i_t^D}{1-\ell} \tag{5.53}$$

Thus r^K is calculated by:

$$\frac{(1-\rho\ell)\, i_t^M - \ell(1-\rho)\, i_t^D}{1-\ell} = \mathbb{E}_t r_{t+1}^K$$

The rest of equations:

$$\begin{aligned}
\widehat{y}_t &= \alpha \widehat{b}_{t-1} + (1-\alpha)\,\widehat{l}_t \\
\widehat{y}_t &= \alpha_c \widehat{c}_t + (1-\alpha_c)\,\widehat{b}_t \\
\widehat{w}_t + \widehat{l}_t &= \widehat{p}_{mt} + \widehat{y}_t \\
\frac{r_t^K - r^K}{1+r^K} &= \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{b}_{t-1}\right) \\
\widehat{w}_t + \widehat{l}_t &= \left(1-\frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + (1+\varphi)\,\widehat{l}_t \\
\widehat{q}_t &= \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \alpha_m \widetilde{m}_t - (1-\alpha_m)\,\widehat{i}_t\right)
\end{aligned} \tag{5.54}$$

Because $\delta = 1$**, we get** $\widehat{k}_{t+1} = \widehat{b}_t = \widehat{i}_t$. Let's combine the first five equations: First, we remove $\widehat{p}_{mt}$ from the last three to obtain:

$$\frac{r_t^K - r^K}{1+r^K} = \alpha_y\left(\left(1-\frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + (\varphi+1)\,\widehat{l}_t - \widehat{i}_{t-1}\right).$$

Next, we remove $\widehat{y}_t$ to obtain $\widehat{y}_t = \alpha_c \widehat{c}_t + (1-\alpha_c)\,\widehat{i}_t = \alpha \widehat{i}_{t-1} + (1-\alpha)\,\widehat{l}_t$. Combining the last two:

$$\frac{i_t^K - r^K}{1+r^K} = \alpha_y\left(\left(1-\frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \frac{\varphi+1}{1-\alpha}\left[\alpha_c \widehat{c}_t + (1-\alpha_c)\,\widehat{i}_t - \alpha \widehat{i}_{t-1}\right] - \widehat{i}_{t-1}\right)$$

Take expectation and combine with the equation for $\mathbb{E}_t r_{t+1}^K$, i.e., $\frac{(1-\rho\ell)\, i_t^M + (\rho\ell-\ell)\, i_t^F}{1-\ell} - r^K = \mathbb{E}_t r_{t+1}^K - r^K$, to obtain

$$\frac{(1-\rho\ell)\, i_t^M - \ell(1-\rho)\, i_t^D}{(1-\ell)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = \alpha_y\left(\left(1-\frac{\eta}{\sigma}\right)\mathbb{E}_t\widehat{q}_{t+1} + \frac{1}{\sigma}\mathbb{E}_t\widehat{c}_{t+1} + \frac{\varphi+1}{1-\alpha}\left[\alpha_c \mathbb{E}_t\widehat{c}_{t+1} + (1-\alpha_c)\,\mathbb{E}_t\widehat{i}_{t+1} - \alpha\widehat{i}_t\right] - \widehat{i}_t\right) \tag{5.55}$$

Unknowns: $i_t^S, \widehat{c}_t, \widehat{i}_t$, $\widehat{q}_t$, i_t^M, with this rule:

$$i_t^M = r^M + \phi_y^M \hat{c}_t + u_t^M \tag{5.56}$$

We need 1 more equations to close the model:

$$\widetilde{m}_t \quad = \quad u_t^m \tag{5.57}$$

B.1 Proof of Proposition 1; Quantity of reserves shock

Now we assume an unexpected one time shock hits the quantity of reserves ($\widetilde{m}_t = \Delta$ and $\widetilde{m} = 0$ for $s > t$). Since this is an unexpected shock and dies in just one period, we can set all expected values to zero, in which case we obtain (unknowns: $i_t^S, \widehat{c}_t, \widehat{i}_t$, $\widehat{q}_t$, i_t^D,i_t^M):

$$\text{Euler equation: } \widehat{c}_t = -\sigma\left(\beta i_t^S + \beta - 1\right) - \left(\sigma - \eta\right)\widehat{q}_t \tag{5.58}$$

$$\frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\alpha_m \Delta - \frac{1}{\eta}(1-\alpha_m)\,\widehat{i}_t \tag{5.59}$$

$$i_t^S - i_t^M = \ell\frac{i_t^M - i_t^D}{1-\ell} \rightarrow i_t^S = \frac{i_t^M - \ell i_t^D}{1-\ell} \Rightarrow i_t^S - i_t^D = \frac{i_t^S - i_t^M}{\ell} \rightarrow i_t^S - i_t^D = \frac{i_t^M - i_t^D}{1-\ell}$$

$$\widehat{q}_t \quad = \quad \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \alpha_m\Delta - (1-\alpha_m)\,\widehat{i}_t\right) \tag{5.60}$$

$$\frac{\left(1-\rho\ell\right)i_t^M - \ell\left(1-\rho\right)i_t^D}{(1-\ell)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t \tag{5.61}$$

$$i_t^M = r^M + \phi_y^M \hat{c}_t \tag{5.62}$$

Let's remove i_t^S, i_t^M, and $\widehat{q}_t$ to simplify the system to:

$$\widehat{c}_t = -\sigma\left(\beta\frac{r^M + \phi_y^M \hat{c}_t - \ell i_t^D}{1-\ell} + \beta - 1\right) - \frac{\sigma-\eta}{\eta}\alpha_{DD}\left(\widehat{c}_t - \alpha_m\Delta - (1-\alpha_m)\widehat{i}_t\right)$$

$$\frac{r^M + \phi_y^M \hat{c}_t - i_t^D}{\left(i^S - i^D\right)(1-\ell)} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\alpha_m\Delta - \frac{1}{\eta}(1-\alpha_m)\widehat{i}_t$$

$$\frac{(1-\rho\ell)\left(r^M + \phi_y^M \hat{c}_t\right) - \ell(1-\rho)\, i_t^D}{(1-\ell)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t$$

Special case of $\sigma = \eta$:

Let's simplify the first equation and solve for $\widehat{c}_t$ from the first equation and then plug it into the second one:

$$BB\widehat{c}_t - \frac{(1-\beta)(1-\ell)\sigma - \sigma\beta r^M}{\sigma\beta\ell} = i_t^D$$

$$\Rightarrow \frac{r^M}{\left(i^S - i^D\right)(1-\ell)} + \left(\frac{\phi_y^M}{\left(i^S - i^D\right)(1-\ell)} - \frac{1}{\eta}\right)\widehat{c}_t - \frac{1}{\left(i^S - i^D\right)(1-\ell)}\left(BB\widehat{c}_t - \frac{(1-\beta)(1-\ell)\sigma - \sigma\beta r^M}{\sigma\beta\ell}\right) - 1 = -\frac{1}{\eta}\alpha_m\Delta - \frac{1}{\eta}(1-\alpha_m)\widehat{i}_t$$

$$\Rightarrow \left(\frac{\phi_y^M - BB}{\left(i^S - i^D\right)(1-\ell)} - \frac{1}{\eta}\right)\widehat{c}_t + \frac{r^M}{\left(i^S - i^D\right)(1-\ell)} + \frac{(1-\beta)(1-\ell)\sigma - \sigma\beta r^M}{\sigma\beta\ell\left(i^S - i^D\right)(1-\ell)} - 1 + \frac{1}{\eta}\alpha_m\Delta = -\frac{1}{\eta}(1-\alpha_m)\widehat{i}_t$$

$$\Rightarrow -\left(\frac{\phi_y^M + \frac{1}{\sigma\beta}}{\left(i^S - i^D\right)\ell} + \frac{1}{\eta}\right)\widehat{c}_t + \frac{1 - \beta\left(1 + r^M\right)}{\beta\ell\left(i^S - i^D\right)} - 1 + \frac{1}{\eta}\alpha_m\Delta = -\frac{1}{\eta}(1-\alpha_m)\widehat{i}_t \qquad (5.63)$$

Now, solve for

$$\frac{(1-\rho\ell)\left(r^M + \phi_y^M \hat{c}_t\right) - \ell(1-\rho)\, i_t^D}{(1-\ell)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\frac{-\eta}{1-\alpha_m}\left[\frac{-1}{\eta}(1-\alpha_m)\widehat{i}_t\right]$$

$$\Rightarrow \frac{\left(1-\rho\ell\right)\left(r^M+\phi_y^M\hat{c}_t\right)-\ell\left(1-\rho\right)\left[BB\widehat{c}_t-\frac{(1-\beta)(1-\ell)\sigma-\sigma\beta r^M}{\sigma\beta\ell}\right]}{(1-\ell)\left(1+r^K\right)}-\frac{r^K}{1+r^K}=-\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t$$

$$=\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\frac{\eta}{1-\alpha_m}\left[-\left(\frac{\phi_y^M+\frac{1}{\sigma\beta}}{\left(i^S-i^D\right)\ell}+\frac{1}{\eta}\right)\widehat{c}_t+\frac{1-\beta\left(1+r^M\right)}{\beta\ell\left(i^S-i^D\right)}-1+\frac{1}{\eta}\alpha_m\Delta\right]$$

$$\Rightarrow\left[\left(1-\rho\ell\right)\phi_y^M-\left(1-\rho\right)\frac{(1-\ell)}{\sigma\beta}-\left(1-\rho\right)\phi_y^M+(1-\ell)\left(1+r^K\right)\frac{\varphi\alpha+1}{1-\alpha}\frac{\eta\alpha_y}{1-\alpha_m}\left(\frac{\phi_y^M+\frac{1}{\sigma\beta}}{\left(i^S-i^D\right)\ell}+\frac{1}{\eta}\right)\right]\widehat{c}_t$$

$$=-\left(1-\rho\ell\right)r^M-\ell\left(1-\rho\right)\frac{\left(1-\beta\right)(1-\ell)\sigma-\sigma\beta r^M}{\sigma\beta\ell}+(1-\ell)r^K+(1-\ell)\left(1+r^K\right)\frac{\varphi\alpha+1}{1-\alpha}\frac{\eta\alpha_y}{1-\alpha_m}\left(\frac{1-\beta\left(1+r^M\right)}{\beta\ell\left(i^S-i^D\right)}-1+\frac{1}{\eta}\alpha_m\Delta\right)$$

We used:

$AA\equiv 1+\frac{\sigma\beta\phi_y^M}{1-\ell}$

$BB=\frac{(1-\ell)AA}{\sigma\beta\ell}\equiv\frac{1-\ell}{\sigma\beta\ell}+\frac{\phi_y^M}{\ell}$

$\phi_y^M-BB=\phi_y^M-\frac{1-\ell}{\sigma\beta\ell}-\frac{\phi_y^M}{\ell}=-\frac{1-\ell}{\ell}\phi_y^M-\frac{1-\ell}{\sigma\beta\ell}=-\frac{1-\ell}{\ell}\left(\phi_y^M+\frac{1}{\sigma\beta}\right)$

$-\ell\left(1-\rho\right)BB=-\left(1-\rho\right)\frac{(1-\ell)}{\sigma\beta}-\left(1-\rho\right)\phi_y^M$

$-\left(1-\rho\right)\frac{(1-\beta)(1-\ell)\sigma-\sigma\beta r^M}{\sigma\beta}=\left(1-\rho\right)r^M-\frac{(1-\rho)(1-\beta)(1-\ell)}{\beta}.$

Simplifying and dividing by $1-\ell$:

$$\widehat{c}_t=\frac{-\rho r^M-\frac{(1-\rho)(1-\beta)}{\beta}+r^K+\left(1+r^K\right)\frac{\varphi\alpha+1}{1-\alpha}\frac{\eta\alpha_y}{1-\alpha_m}\left(\frac{1-\beta(1+r^M)}{\beta\ell(i^S-i^D)}-1+\frac{1}{\eta}\alpha_m\Delta\right)}{\left[\rho\phi_y^M-\frac{1-\rho}{\sigma\beta}+\frac{(\varphi\alpha+1)}{1-\alpha}\frac{\eta\alpha_y(1+r^K)}{1-\alpha_m}\left(\frac{\phi_y^M+\frac{1}{\sigma\beta}}{(i^S-i^D)\ell}+\frac{1}{\eta}\right)\right]} \tag{5.64}$$

$$\Rightarrow\frac{\partial\widehat{c}_t}{\partial\Delta}=\frac{\left(1+r^K\right)\frac{\varphi\alpha+1}{1-\alpha}\frac{\alpha_m}{1-\alpha_m}\alpha_y}{\left[\rho\phi_y^M-\frac{1-\rho}{\sigma\beta}+\frac{(\varphi\alpha+1)}{1-\alpha}\frac{\eta\alpha_y(1+r^K)}{1-\alpha_m}\left(\frac{\phi_y^M+\frac{1}{\sigma\beta}}{(i^S-i^D)\ell}+\frac{1}{\eta}\right)\right]}$$

$$\phi_y^M>\frac{1-\rho}{\eta\beta\rho}\Rightarrow\frac{\partial\widehat{c}_t}{\partial\Delta}>0$$

Noting that $\alpha_y = 1$ and $\sigma = \eta$, we can simplify it to:

$$\Rightarrow \frac{\partial \widehat{c}_t}{\partial \Delta} = \frac{\left(1+r^K\right)\frac{\varphi\alpha+1}{1-\alpha}\frac{\alpha_m}{1-\alpha_m}}{\left[\rho\phi_y^M - \frac{1-\rho}{\eta\beta} + \frac{(\varphi\alpha+1)}{1-\alpha}\frac{(1+r^K)}{1-\alpha_m}\left(\frac{\eta\phi_y^M+\frac{1}{\beta}}{(i^S-i^D)\ell}+1\right)\right]} \tag{5.65}$$

Now we can simplify the equation for investment (eq. 5.63):

$$\Rightarrow \frac{\partial \widehat{i}_t}{\partial \Delta} = \frac{1}{(1-\alpha_m)}\left(\frac{\eta\phi_y^M+\frac{\eta}{\sigma\beta}}{\left(i^S-i^D\right)\ell}+1\right)\frac{\partial \widehat{c}_t}{\partial \Delta} - \frac{\alpha_m}{1-\alpha_m}$$

$$\Rightarrow \frac{(1-\alpha_m)}{\alpha_m}\frac{\partial \widehat{i}_t}{\partial \Delta} = \frac{\left(1+\frac{\eta\phi_y^M+\frac{\eta}{\sigma\beta}}{(i^S-i^D)\ell}\right)\frac{\varphi\alpha+1}{1-\alpha}}{\left[\left(\frac{1-\alpha_m}{1+r^K}\right)\left(\rho\phi_y^M-\frac{1-\rho}{\eta\beta}\right)+\frac{\varphi\alpha+1}{1-\alpha}\left(1+\frac{\eta\phi_y^M+\frac{1}{\beta}}{(i^S-i^D)\ell}\right)\right]} - 1$$

$$\Rightarrow \frac{(1-\alpha_m)}{\alpha_m}\frac{\partial \widehat{i}_t}{\partial \Delta} = -\frac{\left(\frac{1-\alpha_m}{1+r^K}\right)\left(\rho\phi_y^M-\frac{1-\rho}{\eta\beta}\right)}{\left[\frac{\varphi\alpha+1}{1-\alpha}\left(1+\frac{\eta\phi_y^M+\frac{1}{\beta}}{(i^S-i^D)\ell}\right)+\left(\frac{1-\alpha_m}{1+r^K}\right)\left(\rho\phi_y^M-\frac{1-\rho}{\eta\beta}\right)\right]} \tag{5.66}$$

Equations (5.66) and (5.66) are what we want for Proposition 1. The other parts of the proposition are straightforward for consumption and investment. For the interest rates, we can simply use the following:

$$\begin{aligned}
\frac{\partial i_t^M}{\partial \Delta} &= \phi_y^M \frac{\partial \widehat{c}_t}{\partial \Delta} \\
\frac{\partial i_t^D}{\partial \Delta} &= \left(\frac{1-\ell}{\sigma\beta\ell}+\frac{\phi_y^M}{\ell}\right)\frac{\partial \widehat{c}_t}{\partial \Delta} \\
\frac{\partial i_t^S}{\partial \Delta} &= -\frac{1}{\sigma\beta}\frac{\partial \widehat{c}_t}{\partial \Delta} \\
\frac{\partial\left(i_t^M-i_t^D\right)}{\partial \Delta} &= -\left(\frac{1-\ell}{\ell}\right)\left(\frac{1}{\sigma\beta}+\phi_y^M\right)\frac{\partial \widehat{c}_t}{\partial \Delta}
\end{aligned}$$

$$\frac{\partial i_t^K}{\partial \Delta} = \frac{\alpha_m\alpha_y}{(1-\alpha)}\frac{\left[\left(1+r^K\right)(\varphi\alpha+1)\left(\frac{1}{\sigma}+\alpha_c\frac{\varphi+1}{1-\alpha}\right)-(1-\alpha_c)(\varphi+1)\left(\rho\phi_y^M-\frac{1-\rho}{\eta\beta}\right)\right]}{\frac{\varphi\alpha+1}{1-\alpha}\left(1+\frac{\eta\phi_y^M+\frac{1}{\beta}}{(i^S-i^D)\ell}\right)+\left(\frac{1-\alpha_m}{1+r^K}\right)\left(\rho\phi_y^M-\frac{1-\rho}{\eta\beta}\right)}$$

This concludes the proof of Proposition 1.

B.2 Proof of Proposition 2; Reserves interest rate

Now we assume an unexpected one time shock hits the interest on reserves ($u_t^M = \Delta$ and $u_s^M = 0$ for $s > t$), also we assume $\widetilde{m}_t = 0$. Since this is an unexpected shock and dies in just one period, we can set all expected values to zero, in which case we obtain (unknowns: $i_t^S, \widehat{c}_t, \widehat{i}_t, \widehat{q}_t, i_t^D, i_t^M$):

$$\text{Euler equation: } \widehat{c}_t = -\sigma\left(\beta i_t^S + \beta - 1\right) - \left(\sigma - \eta\right)\widehat{q}_t \tag{5.67}$$

$$\frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}(1-\alpha_m)\,\widehat{i}_t \tag{5.68}$$

$$i_t^S - i_t^M = \ell\frac{i_t^M - i_t^D}{1-\ell} \to i_t^S = \frac{i_t^M - \ell i_t^D}{1-\ell} \Rightarrow i_t^S - i_t^D = \frac{i_t^S - i_t^M}{\ell} \to i_t^S - i_t^D = \frac{i_t^M - i_t^D}{1-\ell}$$

$$\widehat{q}_t \quad = \quad \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - (1-\alpha_m)\,\widehat{i}_t\right) \tag{5.69}$$

$$\frac{\left(1-\rho\ell\right)i_t^M - \ell\left(1-\rho\right)i_t^D}{(1-\ell)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t \tag{5.70}$$

$$i_t^M = r^M + \phi_y^M\hat{c}_t + \Delta \tag{5.71}$$

Let's remove i_t^S, i_t^M, and $\widehat{q}_t$ to simplify the system to:

$$\widehat{c}_t = -\sigma\left(\beta\frac{r^M + \phi_y^M\hat{c}_t + \Delta - \ell i_t^D}{1-\ell} + \beta - 1\right) - \frac{\sigma-\eta}{\eta}\alpha_{DD}\left(\widehat{c}_t - (1-\alpha_m)\,\widehat{i}_t\right)$$

$$\frac{r^M + \phi_y^M\hat{c}_t + \Delta - i_t^D}{\left(i^S - i^D\right)(1-\ell)} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}(1-\alpha_m)\,\widehat{i}_t$$

$$\frac{\left(1-\rho\ell\right)\left(r^M + \phi_y^M\hat{c}_t + \Delta\right) - \ell\left(1-\rho\right)i_t^D}{(1-\ell)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t$$

We write the system of equations in matrix form

$$AX = B + C\Delta$$

where

$$X = \begin{bmatrix} \widehat{c}_t \\ \widehat{i}_t \\ i_t^D \end{bmatrix}$$

$$A = \begin{bmatrix} 1 + \sigma\beta\frac{\phi_y^M}{1-\ell} + \frac{\sigma-\eta}{\eta}\alpha_{DD} & -\frac{\sigma-\eta}{\eta}\alpha_{DD}(1-\alpha_m) & -\sigma\left(\beta\frac{\ell}{1-\ell}\right) \\ \frac{\phi_y^M}{(i^S-i^D)(1-\ell)} - \frac{1}{\eta} & \frac{1-\alpha_m}{\eta} & -\frac{1}{(i^S-i^D)(1-\ell)} \\ \frac{(1-\rho\ell)\phi_y^M}{(1-\ell)(1+r^K)} & \alpha_y\frac{\varphi\alpha+1}{1-\alpha} & -\frac{\ell(1-\rho)}{(1-\ell)(1+r^K)} \end{bmatrix}$$

$$B = \begin{bmatrix} -\sigma\left(\beta\frac{r^M}{1-\ell} + \beta - 1\right) \\ 1 - \frac{r^M}{(i^S-i^D)(1-\ell)} \\ \frac{r^K}{1+r^K} - \frac{(1-\rho\ell)r^M}{(1-\ell)(1+r^K)} \end{bmatrix}$$

$$C = \begin{bmatrix} -\sigma\beta\frac{1}{1-\ell} \\ -\frac{1}{(i^S-i^D)(1-\ell)} \\ -\frac{(1-\rho\ell)}{(1-\ell)(1+r^K)} \end{bmatrix}$$

This is the desired matrix format for the linear system.

To calculate $\frac{\partial X}{\partial \Delta}$, we get:

$$\frac{\partial X}{\partial \Delta} = \begin{bmatrix} \frac{\partial \widehat{c}_t}{\partial \Delta} \\ \frac{\partial \widehat{i}_t}{\partial \Delta} \\ \frac{\partial i_t^D}{\partial \Delta} \end{bmatrix} = A^{-1}C$$

$$det(A)A^{-1} = \begin{bmatrix} -\frac{\ell(1-\rho)}{(1-\ell)\left(1+r^K\right)}\frac{1-\alpha m}{\eta} + \frac{1}{\left(i^S-i^D\right)(1-\ell)}\alpha_y\frac{\varphi\alpha+1}{1-\alpha} & \frac{\ell(1-\rho)}{(1-\ell)\left(1+r^K\right)}\left(\frac{\phi_y^M}{\left(i^S-i^D\right)(1-\ell)} - \frac{1}{\eta}\right) - \frac{1}{\left(i^S-i^D\right)(1-\ell)}\frac{(1-\rho\ell)\phi_y^M}{(1-\ell)\left(1+r^K\right)} & \left(\frac{\phi_y^M}{\left(i^S-i^D\right)(1-\ell)} - \frac{1}{\eta}\right)\alpha_y\frac{\varphi\alpha+1}{1-\alpha} - \frac{1-\alpha m}{\eta}\frac{(1-\rho\ell)\cdots}{(1-\ell)(1\cdots} \\ \left(-\frac{\sigma-\eta}{\eta}\alpha_{DD}(1-\alpha m)\frac{\ell(1-\rho)}{(1-\ell)\left(1+r^K\right)} - \sigma\beta\frac{\ell}{1-\ell}\frac{(1-\rho\ell)\phi_y^M}{(1-\ell)\left(1+r^K\right)}\right) & \left(-\frac{\ell(1-\rho)}{(1-\ell)\left(1+r^K\right)}\left(1+\sigma\beta\frac{\phi_y^M}{1-\ell} + \frac{\sigma-\eta}{\eta}\alpha_{DD}\right) + \sigma\beta\frac{\ell}{1-\ell}\frac{(1-\rho\ell)\phi_y^M}{(1-\ell)\left(1+r^K\right)}\right) & \left(-\left(1+\sigma\beta\frac{\phi_y^M}{1-\ell} + \frac{\sigma-\eta}{\eta}\alpha_{DD}\right)\alpha_y\frac{\varphi\alpha+1}{1-\alpha} - \frac{\sigma-\eta}{\eta}\alpha_{DD}(1-\alpha m)\cdots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

$$
\begin{aligned}
-(1-\ell)\,det(A)\frac{\partial\widehat{c}_t}{\partial\Delta} \quad = \quad & -\frac{\ell\left(1-\rho\right)}{(1-\ell)\left(1+r^K\right)}\frac{1-\alpha_m}{\eta}\sigma\beta+\frac{1}{\left(i^S-i^D\right)(1-\ell)}\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\sigma\beta \\
& +\frac{\phi_y^M}{\left(i^S-i^D\right)(1-\ell)}\frac{\ell\left(1-\rho\right)}{(1-\ell)\left(1+r^K\right)}\frac{1}{\left(i^S-i^D\right)} \\
& -\frac{1}{\eta}\frac{\ell\left(1-\rho\right)}{(1-\ell)\left(1+r^K\right)}\frac{1}{\left(i^S-i^D\right)}-\frac{1}{\left(i^S-i^D\right)(1-\ell)}\frac{\left(1-\rho\ell\right)\phi_y^M}{(1-\ell)\left(1+r^K\right)}\frac{1}{\left(i^S-i^D\right)} \\
& +\frac{\phi_y^M}{\left(i^S-i^D\right)(1-\ell)}\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\frac{\left(1-\rho\ell\right)}{\left(1+r^K\right)}-\frac{1}{\eta}\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\frac{\left(1-\rho\ell\right)}{\left(1+r^K\right)}-\frac{1-\alpha_m}{\eta}\frac{\left(1-\rho\ell\right)\phi_y^M}{(1-\ell)\left(1+r^K\right)}\frac{\left(1-\rho\ell\right)}{\left(1+r^K\right)}
\end{aligned}
$$

$$
\begin{aligned}
-(1-\ell)\,det(A)\frac{\partial\widehat{i}_t}{\partial\Delta} \quad = \quad & \left(-\frac{\sigma-\eta}{\eta}\alpha_{DD}(1-\alpha_m)\frac{\ell\left(1-\rho\right)}{(1-\ell)\left(1+r^K\right)}-\sigma\beta\frac{\ell}{1-\ell}\frac{\left(1-\rho\ell\right)\phi_y^M}{(1-\ell)\left(1+r^K\right)}\right)\sigma\beta \\
& +\left(-\frac{\ell\left(1-\rho\right)}{(1-\ell)\left(1+r^K\right)}\left(1+\sigma\beta\frac{\phi_y^M}{1-\ell}+\frac{\sigma-\eta}{\eta}\alpha_{DD}\right)+\sigma\beta\frac{\ell}{1-\ell}\frac{\left(1-\rho\ell\right)\phi_y^M}{(1-\ell)\left(1+r^K\right)}\right)\frac{1}{\left(i^S-i^D\right)} \\
& +\left(-\left(1+\sigma\beta\frac{\phi_y^M}{1-\ell}+\frac{\sigma-\eta}{\eta}\alpha_{DD}\right)\alpha_y\frac{\varphi\alpha+1}{1-\alpha}-\frac{\sigma-\eta}{\eta}\alpha_{DD}(1-\alpha_m)\frac{\left(1-\rho\ell\right)\phi_y^M}{(1-\ell)\left(1+r^K\right)}\right)\frac{\left(1-\rho\ell\right)}{\left(1+r^K\right)}
\end{aligned}
$$

Special case of $\sigma=\eta$

$$
\begin{aligned}
-(1-\ell)\left(1+r^K\right)det(A)\frac{\partial\widehat{i}_t}{\partial\Delta} \quad = \quad & \left[-\sigma\beta+\frac{1}{\left(i^S-i^D\right)}\right]\frac{\ell}{1-\ell}\frac{\left(1-\rho\ell\right)}{(1-\ell)}\sigma\beta\phi_y^M \\
& -\left[\frac{\ell\left(1-\rho\right)}{(1-\ell)}\frac{1}{\left(i^S-i^D\right)}+\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\left(1-\rho\ell\right)\right]\frac{\sigma\beta\phi_y^M}{1-\ell} \\
& -\left[\frac{\ell\left(1-\rho\right)}{(1-\ell)}\frac{1}{\left(i^S-i^D\right)}+\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\left(1-\rho\ell\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
-\left(\frac{1-\ell}{\ell}\right)\left(1+r^K\right)det(A)\frac{\partial\widehat{i}_t}{\partial\Delta} \quad = \quad & \left[\frac{\rho}{\left(i^S-i^D\right)}-\sigma\beta\frac{\left(1-\rho\ell\right)}{(1-\ell)}-\frac{\varphi\alpha+1}{1-\alpha}\left(\frac{1-\rho\ell}{\ell}\right)\right]\frac{\sigma\beta\phi_y^M}{1-\ell} \\
& -\left[\frac{\left(1-\rho\right)}{(1-\ell)}\frac{1}{\left(i^S-i^D\right)}+\frac{\varphi\alpha+1}{1-\alpha}\left(\frac{1-\rho\ell}{\ell}\right)\right]
\end{aligned}
$$

where we also used $\alpha_y=1$.

Assuming that $det(A)<0$, then:

$$\frac{\rho}{\left(i^{S}-i^{D}\right)} < \sigma\beta\frac{\left(1-\rho\ell\right)}{(1-\ell)} + \frac{\varphi\alpha+1}{1-\alpha}\left(\frac{1-\rho\ell}{\ell}\right) \Rightarrow \frac{\partial\widehat{i}_{t}}{\partial\Delta} < 0 \tag{5.72}$$

which brings us to Proposition 2.

C Proof of Proposition 3 and Corollary 1 and associated analysis for the case that CBDC and deposits are perfect substitutes

We are interested in a case where CBDC and deposits are perfect substitutes, i.e., $\upsilon = \infty$ and $\omega_{FD} = \omega_D$, and they are both used in equilibrium. In this case, their interest rate must be equal. We will assess the effects of different shocks in this case: an exogenous shock to CBDC interest rate.

It is similar to a deposit-like CBDC which has been discussed in the literature. Assuming that the CBDC is used in equilibrium, since CBDC can be used in exactly the same set of transactions as deposits with the same importance, we will have:

- If $i^F > i^D$, bank cannot raise deposits. Therefore, $i^D \geq i^F$.
- If $i^D > i^F$, demand for CBDC is zero and CBDC is not used, so we must have $i^D = i^F$.

Steady State Equations:

In this equilibrium, the central bank fixes quantity of reserves and we derive a necessary condition for that. Here, the central bank no longer can decide on i^F.

- Output, consumption and labor: Y, C, L
- Deposits, CBDC and reserves balances: D, F
- Real assets: b
- Rates: i^K, i^D, i^M

As before, the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. We now derive the steady state values:

$$\text{Intermediate good price:} \quad p_m = \frac{\epsilon - 1}{\epsilon},$$
$$\text{Illiquid bond demand:} \quad \beta\left(1 + i^S\right) = 1.$$

$$\text{Deposit and CBDC demand:} \quad \frac{i_t^S - i_t^D}{1 + i_t^S} = \omega_D V_{FD,t}^{\frac{1}{\eta}} = \omega_D \left(\frac{P_t C_t}{D_t + F_t}\right)^{\frac{1}{\eta}} = \frac{i_t^S - i_t^F}{1 + i_t^S} \qquad (5.73)$$

$$\begin{aligned} V_{FD} &\equiv \left(V_D^{-1} + V_F^{-1}\right)^{-1} = \frac{PC}{D+F} \\ Q &\equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \end{aligned}$$

Notice that given the banks' optimality conditions, we must have:

$$\frac{i^S - i^D}{1+i^S} = \frac{i^S - i^F}{1+i^S} = \frac{i^S - i^K}{\left(1+i^S\right)\rho\ell} = \frac{i^S - i^M}{\left(1+i^S\right)\ell}$$

Therefore:

$$\text{Perfect substitution:} \qquad i^D = i^F. \tag{5.74}$$

This gives us a relationship between i^F and i^M:

$$i^M - \ell i^F = (1-\ell)\, i^S$$

In the steady state with zero inflation rate, $i^S = \frac{1}{\beta} - 1$, which implies that i^M is no longer a free variable in the steady state and is given by:

$$i^M = \ell i^F + (1-\ell)\,\frac{1-\beta}{\beta}. \tag{5.75}$$

For a given i^F, the maximum real deposit demand is given by $\frac{D^{max}}{P} = \frac{C}{V_{FD}^{min}}$, so

$$\frac{D^{max}}{P} = \frac{C}{\left(\frac{i^S - i^F}{\omega_D\left(1+i^S\right)}\right)^{\eta}}.$$

Block 1: Given i^F, we can solve for i^K and $i^D = i^F$:

$$\frac{i^S - i^F}{1+i^S} = \frac{i^S - i^K}{\left(1+i^S\right)\rho\ell} \Rightarrow i^K = \rho\ell i^F + \left(1-\rho\ell\right)\frac{1-\beta}{\beta} \tag{5.76}$$

Block 2: Given i^F from policy, we can pin down V_{FD} and Q:

$$V_{FD} = \left(\frac{1}{\omega_D}\frac{i^S - i^F}{1+i^S}\right)^{\eta} \tag{5.77}$$

$$Q \equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \tag{5.78}$$

Still, we don't know D/P and F/P separately.

Moreover, we have

$$b = K = \frac{I}{\delta}.$$

Block 3: Given i^K (from block 1) and Q (from block 2), the following four equations pin down Y, L, C and b (as before):

$$\begin{aligned} Y &= C + \delta K, \\ p_m &= \frac{\epsilon-1}{\epsilon} = \frac{\left(i^K+\delta\right)K}{\alpha Y} \to K = \frac{\epsilon-1}{\epsilon}\frac{\alpha Y}{i^K+\delta}, \\ Y &= AK^{\alpha}L^{1-\alpha}, \\ \frac{\epsilon-1}{\epsilon}Y &= Q^{1-\frac{\eta}{\sigma}}C^{\frac{1}{\sigma}}\psi\frac{L^{1+\varphi}}{1-\alpha}. \end{aligned} \tag{5.79}$$

We can now calculate Y as a function of C and K and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon-1}{\epsilon}\frac{(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}K^{\alpha\frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}}C^{\frac{1}{\sigma}}\psi},$$

$$Y = C + \delta K \to C = Y\left(1 - \frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right). \tag{5.80}$$

Given that K and C are now given in terms of Y, we obtain

$$Y^{\varphi+\frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}}(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}+1}}{\psi Q^{1-\frac{\eta}{\sigma}}\left(1-\frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}}\left(i^K+\delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{5.81}$$

Block 4, a binding leverage constraint gives D/P:

$$\frac{M}{P} + \rho b = \frac{1}{\ell}\frac{D}{P} \tag{5.82}$$

Therefore, $\frac{M}{P}$ **that the CB chooses must satisfy**:

$$\text{Necessary condition for M: } \frac{M}{P} \leq -\rho b + \frac{1}{\ell}\frac{D^{max}}{P} = -\rho b + \frac{1}{\ell}\frac{C}{\left(\frac{i^S - i^F}{\omega_D(1+i^S)}\right)^{\eta}}$$

where $\frac{D^{max}}{P} \equiv \frac{C}{\left(\frac{i^S - i^F}{\omega_D(1+i^S)}\right)^{\eta}}$.

This means that for given i^F, a steady state equilibrium exists iff:

$$\text{Necessary condition: } \rho K < \frac{1}{\ell}\frac{C}{\left(\frac{i^S - i^F}{\omega_D(1+i^S)}\right)^{\eta}} \tag{5.83}$$

Also, $(D+F)/P = CV_{FD}^{-1}$ gives F/D.

At the end, $w = \frac{\epsilon-1}{\epsilon}(1-\alpha)\frac{Y}{L}$ gives w, and $I = \delta K$.

The only difference is that here i^M is endogenous and M/P (real supply of reserves) is exogenous and set by the policy.

Log-linearization for transitional path

Using the following definitions:

$$\begin{aligned}
\tilde{x} &= \hat{x}_t - \hat{p}_t \text{ for } x \in \{d, f, m\}, \\
\alpha_{FD} &\equiv 1, \\
\beta_D &\equiv \frac{V_D^{-(1-\frac{1}{v})}}{V_D^{-(1-\frac{1}{v})} + \frac{\omega_{FD}}{\omega_D} V_F^{-(1-\frac{1}{v})}}, \\
\alpha_m &\equiv \frac{M/P}{M/P + \rho b}, \\
\alpha_c &\equiv \frac{C}{Y}, \\
\alpha_y &\equiv \frac{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K}}{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K} + 1 - \delta}, \\
\alpha_{DD} &\equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}}.
\end{aligned}$$

We now add full price stickiness and full depreciation.

Here is a summary of the log-linearized version of equilibrium conditions:

$$\widehat{\pi}_t = 0$$

$$\text{Euler equation: } \widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right] - \sigma\left(\beta i_t^S + \beta - 1\right) + \left(\sigma - \eta\right)\left(E_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \tag{5.84}$$

$$i_t^D = i_t^F$$

$$\text{Deposit demand: } \quad \tfrac{i_t^S - i_t^F}{i^S - i^D} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \tfrac{\beta_D}{\eta}\widetilde{d} - \tfrac{1-\beta_D}{\eta}\widetilde{f} \tag{5.85}$$

$$\begin{aligned} \text{Bank FOC:} \quad & \ell\left(i_t^S - i_t^F\right) = \left(i_t^S - i_t^M\right) \Rightarrow i_t^S = \frac{i_t^M - \ell i_t^F}{1-\ell} \\ \text{Bank FOC:} \quad & i_t^S - E_t r_{t+1}^K = \rho\ell\left(i_t^S - i_t^F\right) = \rho\ell\frac{i_t^M - i_t^F}{1-\ell} \\ \text{Bank Leverage:} \quad & \widetilde{d}_t = \alpha_m \widetilde{m} + (1-\alpha_m)\,\widehat{i}_t \end{aligned} \tag{5.86}$$

So,

$$i_t^S - i_t^F = \frac{i_t^M - i_t^F}{1-\ell}$$

$$i_t^S - i_t^M = \ell\frac{i_t^M - i_t^F}{1-\ell} \tag{5.87}$$

These imply that $i_t^S > i_t^M \Leftrightarrow i_t^M > i_t^F \Leftrightarrow i_t^S > i_t^F$**.**

Thus r^K is calculated by:

$$\frac{\left(1-\rho\ell\right)i_t^M - \ell\left(1-\rho\right)i_t^F}{1-\ell} = \mathbb{E}_t r_{t+1}^K$$

The rest of equations:

$$\begin{aligned}
\widehat{y}_t &= \alpha \widehat{b}_{t-1} + (1-\alpha)\,\widehat{l}_t \\
\widehat{y}_t &= \alpha_c \widehat{c}_t + (1-\alpha_c)\,\widehat{b}_t \\
\widehat{w}_t + \widehat{l}_t &= \widehat{p}_{mt} + \widehat{y}_t \\
\frac{r_t^K - r^K}{1+r^K} &= \alpha_y \left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{b}_{t-1}\right) \\
\widehat{w}_t + \widehat{l}_t &= \left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \left(1+\varphi\right)\widehat{l}_t \\
\widehat{q}_t &= \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \beta_D \alpha_m \widetilde{m}_t - \beta_D (1-\alpha_m)\,\widehat{i}_t - (1-\beta_D)\,\widetilde{f}_t\right)
\end{aligned} \tag{5.88}$$

Because $\delta = 1$**, we get** $\widehat{k}_{t+1} = \widehat{b}_t = \widehat{i}_t$. Combine the first four equations. We remove $\widehat{p}_{mt}$ from the last two to obtain:

$$\frac{r_t^K - r^K}{1+r^K} = \alpha_y \left(\left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \left(\varphi+1\right)\widehat{l}_t - \widehat{i}_{t-1}\right).$$

We also remove $\widehat{y}_t$ to obtain $\widehat{y}_t = \alpha_c \widehat{c}_t + (1-\alpha_c)\,\widehat{i}_t = \alpha \widehat{i}_{t-1} + (1-\alpha)\,\widehat{l}_t$. Combining the last two:

$$\frac{i_t^K - r^K}{1+r^K} = \alpha_y \left(\left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \frac{\varphi+1}{1-\alpha}\left[\alpha_c \widehat{c}_t + (1-\alpha_c)\,\widehat{i}_t - \alpha \widehat{i}_{t-1}\right] - \widehat{i}_{t-1}\right)$$

Take expectation and combine with equation for $\mathbb{E}_t r_{t+1}^K$, i.e., $\frac{(1-\rho\ell)\,i_t^M + (\rho\ell-\ell)\,i_t^F}{1-\ell} - r^K = \mathbb{E}_t r_{t+1}^K - r^K$, to obtain

$$\frac{\left(1-\rho\ell\right)i_t^M - \ell\left(1-\rho\right)i_t^F}{\left(1-\ell\right)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = \alpha_y \left(\left(1 - \frac{\eta}{\sigma}\right)\mathbb{E}_t \widehat{q}_{t+1} + \frac{1}{\sigma}\mathbb{E}_t \widehat{c}_{t+1} + \frac{\varphi+1}{1-\alpha}\left[\alpha_c \mathbb{E}_t \widehat{c}_{t+1} + (1-\alpha_c)\,\mathbb{E}_t \widehat{i}_{t+1} - \alpha \widehat{i}_t\right] - \widehat{i}_t\right) \tag{5.89}$$

Unknowns: $i_t^S, \widehat{c}_t, \widehat{i}_t, \widehat{q}_t, \widetilde{f}_t, i_t^M$, with this rule:

$$i_t^M = r^M + \phi_y^M \hat{c}_t + u_t^M. \tag{5.90}$$

We need 2 more equations to close the model:

$$\begin{aligned}
i_t^F &= u_t^F \\
\widetilde{m}_t &= u_t^m
\end{aligned}$$

Effects of a one-time unexpected CBDC rate shock

Now we assume an unexpected one time shock hits the interest on CBDC ($i_t^F = \Delta$ and $i_s^F = 0$ for $s > t$), also we assume $\widetilde{m}_t = 0$. Since this is an unexpected shock and dies in just one period, we can set all expected values to zero, in which case we obtain:

$$\text{Euler equation: } \widehat{c}_t = -\sigma\left(\beta i_t^S + \beta - 1\right) - \left(\sigma - \eta\right)\widehat{q}_t \tag{5.91}$$

$$\frac{i_t^S - \Delta}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{\beta_D}{\eta}\left(\alpha_m \widetilde{m} + (1-\alpha_m)\,\widehat{i}_t\right) - \frac{1-\beta_D}{\eta}\widetilde{f} \tag{5.92}$$

$$i_t^S - i_t^M = \ell\frac{i_t^M - \Delta}{1-\ell} \tag{5.93}$$

$$\widehat{q}_t \quad = \quad \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \beta_D\alpha_m\widetilde{m}_t - \beta_D(1-\alpha_m)\,\widehat{i}_t - (1-\beta_D)\widetilde{f}_t\right) \tag{5.94}$$

$$\frac{\left(1-\rho\ell\right)i_t^M - \ell\left(1-\rho\right)\Delta}{(1-\ell)\left(1+r^K\right)} - \frac{r^K}{1+r^K} = -\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t \tag{5.95}$$

$$i_t^M = r^M + \phi_y^M\hat{c}_t. \tag{5.96}$$

Let's remove $\widehat{q}_t$ and i_t^M from the equations to simplify:

$$\widehat{c}_t = -\sigma\left(\beta i_t^S + \beta - 1\right) - \frac{(\sigma-\eta)}{\eta}\alpha_{DD}\left(\widehat{c}_t - \beta_D\alpha_m\widetilde{m}_t - \beta_D(1-\alpha_m)\,\widehat{i}_t - \eta\frac{(1-\beta_D}{\eta})\widetilde{f}_t\right)$$

$$\frac{i_t^S - \Delta}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{\beta_D}{\eta}\left(\alpha_m\widetilde{m} + (1-\alpha_m)\,\widehat{i}_t\right) - \frac{1-\beta_D}{\eta}\widetilde{f}$$

$$i_t^S = \frac{r^M + \phi_y^M\hat{c}_t - \ell\Delta}{1-\ell}$$

$$\frac{(1-\rho\ell)\left(r^M+\phi_y^M\hat{c}_t\right)-\ell(1-\rho)\Delta}{(1-\ell)\left(1+r^K\right)}-\frac{r^K}{1+r^K}=-\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t$$

Now we can simplify to the equation for consumption:

$$\widehat{c}_t=-\sigma\left(\beta\frac{r^M+\phi_y^M\hat{c}_t-\ell\Delta}{1-\ell}+\beta-1\right)\ldots$$

$$-\frac{\sigma-\eta}{\eta}\alpha_{DD}\left(\widehat{c}_t-\beta_D\alpha_m\widetilde{m}_t-\beta_D(1-\alpha_m)\widehat{i}_t+\eta\frac{\frac{r^M+\phi_y^M\hat{c}_t-\ell\Delta}{1-\ell}-\Delta}{i^S-i^D}-\eta-\widehat{c}_t+\beta_D\left(\alpha_m\widetilde{m}+(1-\alpha_m)\widehat{i}_t\right)\right)$$

$$\Rightarrow\widehat{c}_t=-\sigma\left(\beta\frac{r^M+\phi_y^M\hat{c}_t-\ell\Delta}{1-\ell}+\beta-1\right)-(\sigma-\eta)\alpha_{DD}\left(\frac{\frac{r^M+\phi_y^M\hat{c}_t-\ell\Delta}{1-\ell}-\Delta}{i^S-i^D}-1\right)$$

$$\Rightarrow\widehat{c}_t=-\frac{\sigma\left(\beta\frac{r^M-\ell\Delta}{1-\ell}+\beta-1\right)+\alpha_{DD}\left(\frac{\frac{r^M-\ell\Delta}{1-\ell}-\Delta}{i^S-i^D}-1\right)(\sigma-\eta)}{1+\sigma\beta\frac{\phi_y^M}{1-\ell}+\frac{\phi_y^M\alpha_{DD}}{(1-\ell)\left(i^S-i^D\right)}(\sigma-\eta)} \tag{5.97}$$

Finally, solve the investment:

$$\frac{(1-\rho\ell)\left(r^M+\phi_y^M\hat{c}_t\right)-\ell(1-\rho)\Delta}{(1-\ell)\left(1+r^K\right)}-\frac{r^K}{1+r^K}=-\alpha_y\frac{\varphi\alpha+1}{1-\alpha}\widehat{i}_t \tag{5.98}$$

Therefore, consumption's response can be written as:

$$\frac{\partial\widehat{c}_t}{\partial\Delta}=\frac{\sigma\beta\frac{\ell}{1-\ell}+\frac{1}{1-\ell}\frac{\alpha_{DD}}{i^S-i^D}(\sigma-\eta)}{1+\sigma\beta\frac{\phi_y^M}{1-\ell}+\frac{\phi_y^M\alpha_{DD}}{(1-\ell)\left(i^S-i^D\right)}(\sigma-\eta)}>0 \tag{5.99}$$

Note that the relationship between investment and consumption is the same as in the case with no complementarity. The only equation that has changed is that of consumption.

$$-\alpha_y \frac{\varphi\alpha+1}{1-\alpha}\frac{\partial \widehat{i}_t}{\partial \Delta} = \frac{\left(1-\rho\ell\right)\phi_y^M}{(1-\ell)\left(1+r^K\right)} \frac{\sigma\beta\frac{\ell}{1-\ell}+\frac{1}{1-\ell}\frac{\alpha_{DD}}{i^S-i^D}\left(\sigma-\eta\right)}{1+\sigma\beta\frac{\phi_y^M}{1-\ell}+\frac{\phi_y^M\alpha_{DD}}{(1-\ell)\left(i^S-i^D\right)}\left(\sigma-\eta\right)} - \frac{\ell\left(1-\rho\right)}{(1-\ell)\left(1+r^K\right)}$$

$$\frac{\partial \widehat{i}_t}{\partial \Delta} < 0 \Leftrightarrow \frac{\sigma\beta\frac{\ell}{1-\ell}+\frac{1}{1-\ell}\frac{\alpha_{DD}}{i^S-i^D}\left(\sigma-\eta\right)}{\frac{1}{\phi_y^M}+\sigma\beta\frac{1}{1-\ell}+\frac{\alpha_{DD}}{(1-\ell)\left(i^S-i^D\right)}\left(\sigma-\eta\right)} > \frac{\ell\left(1-\rho\right)}{\left(1-\rho\ell\right)}$$

$$\text{Disintermediation: } \frac{\partial \widehat{i}_t}{\partial \Delta} < 0 \Leftrightarrow \phi_y^M > \frac{1-\rho}{\sigma\beta\rho+\frac{1}{\ell}\frac{\alpha_{DD}}{i^S-i^D}\left(\sigma-\eta\right)} \tag{5.100}$$

Therefore, we have Proposition 3.

For the case of no complementarity ($\sigma=\eta$), we can easily determine the investment response:

$$\frac{\partial \widehat{i}_t}{\partial \Delta} = -\left[\frac{\sigma\beta\rho\phi_y^M-\left(1-\rho\right)}{1+\frac{\sigma\beta}{1-\ell}\phi_y^M}\right]\frac{(1-\alpha)\,\ell}{\alpha_y\left(\varphi\alpha+1\right)(1-\ell)\left(1+r^K\right)}.$$

Deriving Corollary 1 is then straight forward.

D Role of Different Model Ingredients

D.1 The Role of Illiquid Bonds for households

We can define interest on illiquid bonds via the following equation:

$$\lambda_t = \beta \boldsymbol{E}\left[\lambda_{t+1}\left(1+i_t^S\right)\right]$$

If we define it this way, the equations should not be changed for households.

In this case, i_t^S can be calculated as follows:

$$1+i_t^S = \frac{\lambda_t}{\beta \boldsymbol{E} \lambda_{t+1}} = \frac{U_{C,t}}{\beta P_t \boldsymbol{E} \frac{U_{C,t+1}}{P_{t+1}}}.$$

This shows that the FOCs for households are unchanged compared with the case where households can hold illiquid bonds.

D.2 Role of Financial Frictions

Banks are subject to

$$N_t = M_t + P_t b_t - D_t - A_t, \tag{5.101}$$

$$D_t \leq \ell(M_t + \rho P_t b_t). \tag{5.102}$$

Banks maximize the expected value of discounted profits minus equity, which is for time t:

$$\begin{aligned} \mathscr{R}_t &= \boldsymbol{E}_t\left\{\bar{\Lambda}_{t,t+1}\Psi_{t+1}\right\} - N_t \\ &= \boldsymbol{E}_t\left\{\bar{\Lambda}_{t,t+1}\left[\begin{array}{c} P_t b_t\left(1+i_{t+1}^K\right) + M_t\left(1+i_t^M\right) \\ -\left(1+i_t^D\right) D_t - \left(1+i_t^S\right) A_t \end{array}\right]\right\} - N_t, \end{aligned}$$

where $\bar{\Lambda}_{t,t+1} \equiv \beta U_{C,t+1}/(\pi_{t+1} U_{C,t})$ is the nominal stochastic discount factor.

Bank's Optimality Conditions

We impose $N = 0$ then write the Lagrangian for the maximization problem as

$$\mathbb{E}_t \left\{ \bar{\Lambda}_{t,t+1} \left[\begin{array}{c} P_t b_t \left(1 + i^K_{t+1}\right) + M_t \left(1 + i^M_t\right) \\ -\left(1 + i^D_t\right) D_t - \left(1 + i^S_t\right) A_t \end{array} \right] \right\}$$
$$+\bar{\lambda}_t \left(-D_t + \ell M_t + \ell \rho P_t b_t\right),$$
$$-\varpi \left(-A_t - D_t + M_t + P_t b_t\right)$$

where $\bar{\lambda}_t$ and ϖ are the Lagrangian multiplier associated with the leverage constraint and the balance sheet identity.

We then manipulate the optimality conditions to obtain

$$\text{FOC A} : -\mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^S_t\right)\right\} + \varpi \leq 0 \text{ w. ineq. only if } A = 0.$$

$$\text{FOC D} : -\mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^D_t\right)\right\} - \bar{\lambda}_t + \varpi \leq 0 \text{ w. ineq. only if } D = 0.$$

$$\text{FOC b} : \mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^K_{t+1}\right)\right\} + \ell \rho \bar{\lambda}_t - \varpi \leq 0 \text{ w. ineq. only if } b = 0.$$

$$\text{FOC M} : \mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^M_t\right)\right\} + \ell \bar{\lambda}_t - \varpi \leq 0 \text{ w. ineq. only if } M = 0.$$

.... **Assume $\bar{\lambda}_t$ and ϖ are both strictly positive:**

Note that $b > 0$ and $d > 0$, so we have:

$$\mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^D_t\right)\right\} + \bar{\lambda}_t = \varpi = \mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^K_{t+1}\right)\right\} + \ell \rho \bar{\lambda}_t \tag{5.103}$$

Assuming that reserves are also used:

$$\mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^M_t\right)\right\} + \ell \bar{\lambda}_t = \varpi \tag{5.104}$$

Assuming that illiquid bonds are also used, i.e., $A > 0$:

$$\mathbb{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^S_t\right)\right\} = \varpi \tag{5.105}$$

Eventually:

$$\boldsymbol{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(i_t^S - i_t^D\right)\right\} = \frac{E_t\left\{\bar{\Lambda}_{t,t+1}\left(i_t^S - i_t^M\right)\right\}}{\ell} = \frac{E_t\left\{\bar{\Lambda}_{t,t+1}\left(i_t^S - i_{t+1}^K\right)\right\}}{\ell\rho} = \bar{\lambda}_t,$$

$$\Rightarrow \frac{i_t^S - i_t^D}{1 + i_t^S} = \frac{i_t^S - i_t^M}{\ell\left(1 + i_t^S\right)} = \frac{i_t^S - \frac{\boldsymbol{E}_t\left(\bar{\Lambda}_{t+1} i_{t+1}^K\right)}{\boldsymbol{E}_t \bar{\Lambda}_{t+1}}}{\ell\rho\left(1 + i_t^S\right)} = \bar{\lambda}_t. \tag{5.106}$$

From household's condition, we know $\mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^S\right)\right\} = 1$, so $\varpi = 1$.

If illiquid bonds are not used, i.e., $A = 0$. In this case: $M_t + P_t b_t = D_t$, which means $D_t \leq \ell(M_t + \rho P_t b_t)$ cannot hold given that D_t, M_t, and b_t) are all positive.

.... **Assume $\bar{\lambda}_t = 0$ (no Financial Friction) and ϖ is strictly positive:**

Now:

$$\text{FOC A} : -\mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^S\right)\right\} + \varpi \leq 0 \text{ w. ineq. only if } A = 0.$$

$$\text{FOC D} : -\mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^D\right)\right\} + \varpi \leq 0 \text{ w. ineq. only if } D = 0.$$

$$\text{FOC b} : \mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_{t+1}^K\right)\right\} - \varpi \leq 0 \text{ w. ineq. only if } b = 0.$$

$$\text{FOC M} : \mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^M\right)\right\} - \varpi \leq 0 \text{ w. ineq. only if } M = 0.$$

Note that $b > 0$ and $d > 0$, so we have:

$$\mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^D\right)\right\} = \varpi = \mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_{t+1}^K\right)\right\} \tag{5.107}$$

From households' problem, we know $i_t^S > i_t^D$, we should have $A = 0$; otherwise we have $A > 0$ implying that:

$$\varpi = \mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^S\right)\right\} > \mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^D\right)\right\} = \varpi$$

which is a contradiction. Therefore, $A = 0$ thus:

$$M_t + P_t b_t = D_t \tag{5.108}$$

If $M > 0$, then $\mathbb{E}_t\left\{\bar{\Lambda}_{t,t+1}\left(1 + i_t^M\right)\right\} = \varpi$, so

$$\text{Bank FOC 1:} \quad i_t^M = i_t^D \tag{5.109}$$

$$\text{Bank FOC 2:} \quad i_t^M = E_t r_{t+1}^K + \left(1 + r^K\right) E_t \widehat{\pi}_{t+1} \tag{5.110}$$

$$\text{Balance sheet identity:} \quad \widetilde{d} = \alpha_m \widetilde{m} + (1 - \alpha_m) \widehat{b}_t \tag{5.111}$$

where

$$\alpha_m \equiv \frac{M/P}{M/P + b}. \tag{5.112}$$

The third equation is the same as (5.153), but **notice that there is no ρ in the denominator of α_m.**

Altogether, we have

$$\begin{aligned} i_t^{M*} \leq i_t^M &= i_{t+1}^K = i_t^D < i_t^S, \text{ and } m_t > 0, d_t = m_t + b_t, \\ i_t^M < i_t^{M*} &\leq i_{t+1}^K = i_t^D < i_t^S, \text{ and } m_t = 0, d_t = b_t, \end{aligned}$$

and i_t^{M*} is given by: $b_t\left(i_t^M\right) - d_t\left(i_t^M\right) = 0$.

We focus on $i_t^M > i_t^{M*}$.

Steady state analysis for the case with no financial frictions

Note that the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. We now derive the steady state values:

$$\begin{aligned} \text{Intermediate good price:} &\quad p_m = \frac{\epsilon - 1}{\epsilon}, \\ \text{Illiquid bond demand:} &\quad \beta\left(1 + i^S\right) = 1. \end{aligned}$$

Block 1: Given i^M, we can solve for i^K and i^D:

$$i^D = i^M = i^K. \tag{5.113}$$

Block 2: Given i^D from block 1 and i^F from policy, we can pin down V_D and V_F:

$$\text{Deposit demand:} \quad \frac{i^S - i^D}{1+i^S} = \omega_D V_D^{\frac{1}{\upsilon}} V_{FD}^{-\frac{1}{\upsilon}+\frac{1}{\eta}}, \tag{5.114}$$

$$\text{CBDC demand:} \quad \frac{i^S - i^F}{1+i^S} = \omega_{FD} V_F^{\frac{1}{\upsilon}} V_{FD}^{-\frac{1}{\upsilon}+\frac{1}{\eta}}. \tag{5.115}$$

Note that V_{FD} is a function of V_J's. We can then calculate Q:

$$V_{FD} \equiv \left(V_D^{-(1-\frac{1}{\upsilon})} + \frac{\omega_{FD}}{\omega_D} V_F^{-(1-\frac{1}{\upsilon})} \right)^{\frac{-1}{1-\frac{1}{\upsilon}}}$$

$$Q \quad \equiv \quad \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} \right)^{\frac{1}{1-\eta}}$$

Moreover, we have

$$b = K = \frac{I}{\delta}.$$

Block 3: Given i^K (from block 1) and Q (from block 2), the following four equations pin down Y, L, C and b:

$$\begin{aligned} Y &= C + \delta K, \\ p_m &= \frac{\epsilon - 1}{\epsilon} = \frac{\left(i^K + \delta\right) K}{\alpha Y} \to K = \frac{\epsilon - 1}{\epsilon} \frac{\alpha Y}{i^K + \delta}, \\ Y &= A K^{\alpha} L^{1-\alpha}, \\ \frac{\epsilon - 1}{\epsilon} Y &= Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi \frac{L^{1+\varphi}}{1-\alpha}. \end{aligned} \tag{5.116}$$

We can now calculate Y as a function of C and K and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon - 1}{\epsilon} \frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} K^{\alpha \frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi},$$

$$Y = C + \delta K \to C = Y \left(1 - \frac{\epsilon - 1}{\epsilon} \frac{\alpha \delta}{i^K + \delta} \right). \tag{5.117}$$

Given that K and C are now given in terms of Y from Equation (5.116) and Equation (5.117), we obtain

$$Y^{\varphi+\frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}}(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}+1}}{\psi Q^{1-\frac{\eta}{\sigma}}\left(1-\frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}}\left(i^K+\delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{5.118}$$

Block 4, Without leverage constraint, we derive M/P from the balance sheet identity:

$$\frac{D}{P} = \frac{M}{P} + b, \tag{5.119}$$

where $b = K$ comes from block 3 and $D/P = CV_D^{-1}$ comes from block 1 (for V_D) and block 3 (for C).

At the end, $w = \frac{\epsilon-1}{\epsilon}(1-\alpha)\frac{Y}{L}$ gives w, and $I = \delta K$. Also, $F/P = CV_F^{-1}$ gives F.

E Robustness check: Analytical results with no complementarity, no financial frictions, and flexible prices

In this section, we study the model with neutral money: $\eta = \sigma$, full depreciation, $\delta = 1$, flexible prices, $\kappa = 0$, and no financial friction. We also assume exogenous $\widetilde{m}$. First, we keep CBDC in the model and show equations with CBDC but later on we remove it to get some analytical results.

Because $\delta = 1$, we have $\widehat{k}_{t+1} = \widehat{b}_t = \widehat{i}_t$, and $\widehat{w}_t + \widehat{l}_t = \frac{1}{\sigma}\widehat{c}_t + (\varphi+1)\,\widehat{l}_t = \widehat{p}_{mt} + \widehat{y}_t$. Using the following definitions:

$$
\begin{aligned}
\widetilde{x} &= \widehat{x}_t - \widehat{p}_t \text{ for } x \in \{d, f, m\}, \\
\alpha_{FD} &\equiv 1, \\
\beta_D &\equiv \frac{V_D^{-(1-\frac{1}{v})}}{V_D^{-(1-\frac{1}{v})} + \frac{\omega_{FD}}{\omega_D} V_F^{-\left(1-\frac{1}{v}\right)}}, \\
\alpha_c &\equiv \frac{C}{Y}, \\
\alpha_y &\equiv \frac{\alpha \frac{\epsilon-1}{\epsilon}\frac{Y}{K}}{\alpha \frac{\epsilon-1}{\epsilon}\frac{Y}{K} + 1 - \delta} = 1, \\
\alpha_{DD} &\equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}}.
\end{aligned}
$$

Here is a summary of the log-linearized version of equilibrium conditions: $\eta = \sigma$, we have:

$$
\text{Euler equation: } \widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right] - \sigma\left(\beta i_t^S - E_t\left[\widehat{\pi}_{t+1}\right] + \beta - 1\right) \tag{5.120}
$$

$$
\text{Deposit demand: } \frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \left(\frac{1-\beta_D}{v} + \frac{\beta_D}{\eta}\right)\widetilde{d}_t - \left(-\frac{1}{v} + \frac{1}{\eta}\right)(1-\beta_D)\widetilde{f}_t \tag{5.121}
$$

$$
\text{CBDC demand: } \frac{i_t^S - i_t^F}{i^S - i^F} - 1 = \frac{1}{\eta}\widehat{c}_t - \left(\frac{1}{\eta} - \frac{1}{v}\right)\beta_D \widetilde{d}_t - \left(\frac{1-\beta_D}{\eta} + \frac{\beta_D}{v}\right)\widetilde{f}_t \tag{5.122}
$$

$$\begin{aligned}
\text{Bank FOC 1:} &\qquad i_t^M = i_t^D \\
\text{Bank FOC 2:} &\qquad i_t^M = E_t r_{t+1}^K + \left(1 + r^K\right) E_t \widehat{\pi}_{t+1} \\
\text{Balance sheet identity:} &\qquad \widetilde{d} = \alpha_m \widetilde{m} + (1 - \alpha_m) \widehat{b}_t
\end{aligned}$$

where

$$\alpha_m \equiv \frac{M/P}{M/P + b}.$$

The third equation is the same as (5.153), but **notice that there is no ρ in the denominator of α_m. Also, $i_t^{M*} \leq i_t^M$.**

$$\widehat{p}_{mt} = 0$$

The rest of equations:

$$\begin{aligned}
\widehat{y}_t &= \alpha \widehat{b}_{t-1} + (1 - \alpha) \widehat{l}_t \\
\widehat{y}_t &= \alpha_c \widehat{c}_t + (1 - \alpha_c) \widehat{b}_t \\
\frac{1}{\sigma} \widehat{c}_t + \left(\varphi + 1\right) \widehat{l}_t &= \widehat{p}_{mt} + \widehat{y}_t \Rightarrow \left(\varphi + 1\right) \widehat{l}_t = \widehat{y}_t - \frac{1}{\sigma} \widehat{c}_t \\
\frac{r_t^K - r^K}{1 + r^K} &= \alpha_y \left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{b}_{t-1}\right)
\end{aligned}$$

Therefore:

$$\widehat{y}_t = \alpha \widehat{b}_{t-1} + \frac{(1-\alpha)}{\left(\varphi + 1\right)} \widehat{y}_t - \frac{(1-\alpha)}{\left(\varphi + 1\right)} \frac{1}{\sigma} \widehat{c}_t \Rightarrow \widehat{y}_t = \frac{\left(\varphi + 1\right)}{\left(\varphi + \alpha\right)} \alpha \widehat{b}_{t-1} - \frac{(1-\alpha)}{\left(\varphi + \alpha\right)} \frac{1}{\sigma} \widehat{c}_t \tag{5.123}$$

$$\widehat{y}_t = \alpha_c \widehat{c}_t + (1 - \alpha_c) \widehat{b}_t \Rightarrow \frac{\left(\varphi + 1\right)}{\left(\varphi + \alpha\right)} \alpha \widehat{b}_{t-1} - (1 - \alpha_c) \widehat{b}_t = \left(\alpha_c + \frac{(1-\alpha)}{\left(\varphi + \alpha\right)} \frac{1}{\sigma}\right) \widehat{c}_t \tag{5.124}$$

$$\frac{r_t^K - r^K}{1 + r^K} = \alpha_y \left(\widehat{y}_t - \widehat{b}_{t-1}\right) = \alpha_y \left(\left(\frac{\left(\varphi + 1\right)}{\left(\varphi + \alpha\right)} \alpha - 1\right) \widehat{b}_{t-1} - \frac{(1-\alpha)}{\left(\varphi + \alpha\right)} \frac{1}{\sigma} \widehat{c}_t\right)$$

$$= \alpha_y \left(\frac{\left(\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma}\right) \left(\frac{(\varphi+1)}{(\varphi+\alpha)} \alpha - 1\right) \widehat{b}_{t-1}}{\left(\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma}\right)} - \frac{\frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} \left(\frac{(\varphi+1)}{(\varphi+\alpha)} \alpha \widehat{b}_{t-1} - (1 - \alpha_c) \widehat{b}_t\right)}{\left(\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma}\right)} \right)$$

$$\Rightarrow \frac{r_t^K - r^K}{1+r^K} = \alpha_y \left(\frac{\left[\left(\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} \right) \left(\frac{(\varphi+1)}{(\varphi+\alpha)} \alpha - 1 \right) - \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} \frac{(\varphi+1)}{(\varphi+\alpha)} \alpha \right] \widehat{b}_{t-1} - \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} (1-\alpha_c) \widehat{b}_t}{\left(\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} \right)} \right)$$

$$\frac{r_{t+1}^K - r^K}{1+r^K} = \alpha_y \left(\frac{\left[\alpha_c \frac{(\varphi+1)}{(\varphi+\alpha)} \alpha - \alpha_c - \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} \right] \widehat{b}_t - \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} (1-\alpha_c) \widehat{b}_{t+1}}{\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma}} \right) \tag{5.125}$$

Bank block:

$$\text{Bank FOC 1:} \quad i_t^M = i_t^D \tag{5.126}$$

$$\begin{aligned} \text{Bank FOC 2:} \quad i_t^M &= \left(1+r^K\right) \frac{E_t r_{t+1}^K - r^K}{\left(1+r^K\right)} + r_K + \left(1+r^K\right) E_t \widehat{\pi}_{t+1} \\ &= \left(1+r^K\right) \alpha_y \left(\frac{\left[\alpha_c \frac{(\varphi+1)}{(\varphi+\alpha)} \alpha - \alpha_c - \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} \right] \widehat{b}_t - \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} (1-\alpha_c) \mathbb{E}_t \widehat{b}_{t+1}}{\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma}} \right) \\ &\quad + r_K + \left(1+r^K\right) \mathbb{E}_t \widehat{\pi}_{t+1} \end{aligned} \tag{5.127}$$

$$\text{Balance sheet identity:} \quad \widetilde{d}_t = \alpha_m \widetilde{m}_t + (1-\alpha_m) \widehat{b}_t \tag{5.128}$$

We take into account the zero-interest CBDC: $i_t^F = 0$.

Add a Taylor rule for i^M as a function of $\widehat{c}_t$:

$$i_t^M = r^M + \phi_\pi^M \widehat{\pi}_t + \phi_y^M \widehat{c}_t + u_t^M.$$

Now we assume an unexpected one time shock hits the quantity of reserves ($\widetilde{m}_t = \Delta$ and $\widetilde{m} = 0$ for $s > t$). Since this is an unexpected shock and dies in just one period, we can set all expected values to zero, in which case we obtain (unknowns: $i_t^S, \widehat{c}_t, \widehat{b}_t$, $i_t^D, i_t^M, \widehat{\pi}_t, \widetilde{d}_t \widetilde{f}_t$):

Put equations together:

$$\text{Supply side}: \frac{(\varphi+1)}{(\varphi+\alpha)} \alpha \widehat{b}_{t-1} - (1-\alpha_c) \widehat{b}_t = \left(\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)} \frac{1}{\sigma} \right) \widehat{c}_t$$

$$\text{Euler equation: } \widehat{c}_t = -\sigma \left(\beta i_t^S + \beta - 1 \right)$$

$$\text{Deposit demand:} \quad \frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \left(\frac{1-\beta_D}{\upsilon} + \frac{\beta_D}{\eta}\right)\widetilde{d}_t - \left(-\frac{1}{\upsilon} + \frac{1}{\eta}\right)(1-\beta_D)\widetilde{f}_t$$

$$\text{CBDC demand:} \quad \frac{i_t^S - i_t^F}{i^S - i^F} - 1 = \frac{1}{\eta}\widehat{c}_t - \left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)\beta_D\widetilde{d}_t - \left(\frac{1-\beta_D}{\eta} + \frac{\beta_D}{\upsilon}\right)\widetilde{f}_t$$

$$\text{Bank FOC 1:} \quad i_t^M = i_t^D$$

$$\text{Bank FOC 2:} \quad i_t^M = \left(1 + r^K\right)\alpha_y \frac{\left[\alpha_c \frac{(\varphi+1)}{(\varphi+\alpha)}\alpha - \alpha_c - \frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}\widehat{b}_t + r_K$$

$$\text{Balance sheet identity:} \quad \widetilde{d}_t = \alpha_m \Delta + (1-\alpha_m)\widehat{b}_t$$

$$\text{Taylor rule:} i_t^M = r^M + \phi_\pi^M \widehat{\pi}_t + \phi_y^M \widehat{c}_t$$

Special case: No CBDC

Remove i_t^D and write:

$$\text{Supply side:} \frac{(\varphi+1)}{(\varphi+\alpha)}\alpha\widehat{b}_{t-1} - (1-\alpha_c)\widehat{b}_t = \left(\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\widehat{c}_t$$

$$\text{Euler equation: } \widehat{c}_t = -\sigma\left(\beta i_t^S + \beta - 1\right)$$

$$\text{Deposit demand:} \quad \frac{i_t^S - i_t^M}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{d}_t$$

$$\text{Bank FOC 2:} \quad i_t^M = \left(1 + r^K\right)\alpha_y \frac{\left[\alpha_c \frac{(\varphi+1)}{(\varphi+\alpha)}\alpha - \alpha_c - \frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c + \frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}\widehat{b}_t + r_K$$

$$\text{Balance sheet identity:} \quad \widetilde{d}_t = \alpha_m \Delta + (1-\alpha_m)\widehat{b}_t$$

Finally, we get the following 4 equations:

$$\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha\widehat{b}_{t-1}-(1-\alpha_c)\,\widehat{b}_t=\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\widehat{c}_t$$

$$\text{Euler equation: } \widehat{c}_t=-\sigma\left(\beta i_t^S+\beta-1\right)$$

$$\text{Deposit demand:}\quad i_t^S-\left(1+r^K\right)\alpha_y\frac{\left[\alpha_c\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha-\alpha_c-\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}\widehat{b}_t-r_K-\left(i^S-i^D\right)=\left(i^S-i^D\right)\frac{1}{\eta}\widehat{c}_t-\left(i^S-i^D\right)\frac{1}{\eta}\widetilde{d}_t$$

$$\text{Balance sheet identity:}\qquad \widetilde{d}_t=\alpha_m\Delta+(1-\alpha_m)\,\widehat{b}_t$$

Eventually, we get the following equations in $\widehat{b}_t$ and i_t^S:

$$\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha\widehat{b}_{t-1}-(1-\alpha_c)\,\widehat{b}_t=-\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\sigma\left(\beta i_t^S+\beta-1\right)\tag{5.129}$$

$$\begin{aligned}&i_t^S-\left(1+r^K\right)\alpha_y\frac{\left[\alpha_c\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha-\alpha_c-\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}\widehat{b}_t-r_K-\left(i^S-i^D\right)\\&=\left(i^S-i^D\right)\frac{1}{\eta}\sigma\left(\beta i_t^S+\beta-1\right)-\left(i^S-i^D\right)\frac{1}{\eta}\left(\alpha_m\Delta+(1-\alpha_m)\,\widehat{b}_t\right)\end{aligned}$$

$$\begin{aligned}&\Rightarrow i_t^S\left(1-\left(i^S-i^D\right)\beta\right)=\\&\left[\left(1+r^K\right)\alpha_y\frac{\left[\alpha_c\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha-\alpha_c-\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}+\left(i^S-i^D\right)\frac{1}{\eta}(1-\alpha_m)\right]\widehat{b}_t-\left(i^S-i^D\right)\frac{1}{\eta}\alpha_m\Delta+\left[r_K+\left(i^S-i^D\right)\beta\right]\end{aligned}$$

These two equations determine the evolution of capital and illiquid interest rate.

Put everything back to the first equation:

$$\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha\widehat{b}_{t-1} = (1-\alpha_c)\,\widehat{b}_t...$$

$$...-\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\sigma\Bigg(\frac{\beta}{1-\left(i^S-i^D\right)\beta}\Bigg(\Bigg[\left(1+r^K\right)\alpha_y\frac{\left[\alpha_c\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha-\alpha_c-\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}+\left(i^S-i^D\right)\frac{1}{\eta}(1-\alpha_m)\Bigg]\widehat{b}_t-\left(i^S-i^D\right)\frac{1}{\eta}\alpha_m\Delta+\left[r_K+\left(i^S-i^D\right)\beta\right]\Bigg)+\beta-1\Bigg)$$

$$-\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha\widehat{b}_{t-1}=\left[-(1-\alpha_c)+\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\sigma\left[\left(1+r^K\right)\alpha_y\frac{\left[\alpha_c\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha-\alpha_c-\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}+\left(i^S-i^D\right)\frac{1}{\eta}(1-\alpha_m)\right]\frac{\beta}{1-\left(i^S-i^D\right)\beta}\right]\widehat{b}_t...$$

$$...-\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\sigma\frac{\beta}{1-\left(i^S-i^D\right)\beta}\left(i^S-i^D\right)\frac{1}{\eta}\alpha_m\Delta+\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\sigma\left[\frac{\beta}{1-\left(i^S-i^D\right)\beta}\left[r_K+\left(i^S-i^D\right)\beta\right]+\beta-1\right]$$

Given that we began from the SS, so $\widehat{b}_0=0$, we get:

$$\widehat{b}_1=\frac{\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\frac{\beta\left(i^S-i^D\right)}{1-\left(i^S-i^D\right)\beta}\alpha_m}{\left(\sigma\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\right)\left[\left(1+r^K\right)\alpha_y\frac{\left[\alpha_c\frac{(\varphi+1)}{(\varphi+\alpha)}\alpha-\alpha_c-\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right]}{\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}}+\left(i^S-i^D\right)\frac{1}{\eta}(1-\alpha_m)\right]\frac{\beta}{1-\left(i^S-i^D\right)\beta}-(1-\alpha_c)}\Delta \tag{5.130}$$

Note that $-\left(\alpha_c+\frac{(1-\alpha)}{(\varphi+\alpha)}\frac{1}{\sigma}\right)\sigma\left[\frac{\beta}{1-\left(i^S-i^D\right)\beta}\left[r_K+\left(i^S-i^D\right)\beta\right]+\beta-1\right]=0$ so it is dropped from the equation.

Assuming that the denominator is negative, **this implies that the response of capital and investment is negative.**

Finally, note that we don't need Taylor rule, because it only affects evolution of inflation! That is, MP interest rate is ineffective with flexible prices, which is what we know. However, our results show that quantities matter even without financial frictions and without complementarity.

F The Model with Cash

F.1 Derivation of HH's optimality conditions in the full model with cash

Modified utility function with cash is given by:[16]

$$
\begin{aligned}
& U\left(C_t, \frac{D_t}{P_t}, \frac{E_t}{P_t}, \frac{F_t}{P_t}, H_t\right) \\
= & \frac{1}{1-\frac{1}{\sigma}} \begin{pmatrix} C_t^{1-\frac{1}{\eta}} \\ +\omega_D\left((D_t/P_t)^{1-\frac{1}{\upsilon}} + \frac{\omega_{FD}}{\omega_D}(F_t/P_t)^{1-\frac{1}{\upsilon}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\upsilon}}} \\ +\omega_E\left((E_t/P_t)^{1-\frac{1}{\varrho}} + \frac{\omega_{FE}}{\omega_E}(F_t/P_t)^{1-\frac{1}{\varrho}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\varrho}}} \end{pmatrix}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}} - \frac{\psi}{1+\varphi} H_t^{1+\varphi}
\end{aligned}
$$

Optimality conditions:

$$
\begin{aligned}
C &: \frac{U_{C,t}}{P_t} = \lambda_t \\
J &: \frac{U_{J,t}}{P_t} = \lambda_t - \beta \boldsymbol{E} \lambda_{t+1}\left(1+i_t^J\right) \text{ for } J \in \{D, E, F\} \\
S &: \lambda_t = \beta \boldsymbol{E}_t \lambda_{t+1}\left(1+i_t^S\right) \\
\overset{J,S}{\Longrightarrow} \frac{U_{J,t}}{P_t} &= \lambda_t - \beta \boldsymbol{E} \lambda_{t+1}\left(1+i_t^S - i_t^S + i_t^J\right) = \beta \boldsymbol{E} \lambda_{t+1}\left(i_t^S - i_t^J\right) = \lambda_t \frac{i_t^S - i_t^J}{1+i_t^S} \\
H &: U_{H,t} = -\lambda_t W_t
\end{aligned}
$$

The FOCs can be summarized as

$$
\frac{U_C}{P_t} = \frac{U_D}{P_t}\frac{1+i^S}{i^S-i^D} = \frac{U_E}{P_t}\frac{1+i^S}{i^S} = \frac{U_F}{P_t}\frac{1+i^S}{i^S-i^F} = \frac{-U_H}{W_t} = \lambda_t = \beta \boldsymbol{E}_t\left[\lambda_{t+1}(1+i_t^S)\right]
$$

$$
1 = \frac{U_D}{U_C}\frac{1+i^S}{i^S-i^D} = \frac{U_E}{U_C}\frac{1+i^S}{i^S} = \frac{U_F}{U_C}\frac{1+i^S}{i^S-i^F} = \frac{1}{U_C}\frac{-U_H}{W_t/P_t} = \frac{\lambda_t P_t}{U_C}
$$

[16] A cash-like CBDC is a perfect substitute for cash, which can be modeled by $\varrho = \infty$. A universal CBDC (as in Chiu and Davoodalhosseini (2023)) is a perfect substitute for cash and deposits: $\upsilon = \varrho = \infty$.

The money demand function for deposits and cash are given by

$$\begin{aligned}\frac{i^S - i^D}{1+i^S} &= \omega_D \left(\frac{P_t C_t}{D_t}\right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}} \\ \frac{i^S}{1+i^S} &= \omega_E \left(\frac{P_t C_t}{E_t}\right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}}\end{aligned}$$

where $Q_{D,t} \equiv \left(\left(\frac{D_t}{P_t C_t}\right)^{1-\frac{1}{\upsilon}} + \frac{\omega_{FD}}{\omega_D}\left(\frac{F_t}{P_t C_t}\right)^{1-\frac{1}{\upsilon}}\right)^{\frac{1}{1-\frac{1}{\upsilon}}}$ and $Q_{E,t} \equiv \left(\left(\frac{E_t}{P_t C_t}\right)^{1-\frac{1}{\varrho}} + \frac{\omega_{FE}}{\omega_E}\left(\frac{F_t}{P_t C_t}\right)^{1-\frac{1}{\varrho}}\right)^{\frac{1}{1-\frac{1}{\varrho}}}$.

FOC for CBDC:

$$\begin{aligned}1 &= \left(\frac{\omega_{FD}}{\omega_D}\left(\frac{D_t}{F_t}\right)^{\frac{1}{\upsilon}}\frac{U_D}{U_C} + \frac{\omega_{FE}}{\omega_E}\left(\frac{E_t}{F_t}\right)^{\frac{1}{\varrho}}\frac{U_E}{U_C}\right)\frac{1+i^S}{i^S - i^F} \\ \text{so } i^S - i^F &= \frac{\omega_{FD}}{\omega_D}\left(\frac{D_t}{F_t}\right)^{\frac{1}{\upsilon}}\left(i^S - i^D\right) + \frac{\omega_{FE}}{\omega_E}\left(\frac{E_t}{F_t}\right)^{\frac{1}{\varrho}} i^S \\ \text{or } \frac{i^S - i^F}{1+i^S} &= \omega_{FD}\left(\frac{P_t C_t}{F_t}\right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}} + \omega_{FE}\left(\frac{P_t C_t}{F_t}\right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}} \qquad (5.131)\end{aligned}$$

For labor:

$$\begin{aligned}\psi H_t^{\varphi} &= \lambda_t W_t = \frac{W_t}{P_t} C_t^{-\frac{1}{\eta}} XX^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}-1} = \frac{W_t}{P_t} C_t^{-\frac{1}{\eta}} \left(\frac{Q_t}{C_t^{\frac{1}{\eta}}}\right)^{(1-\eta)\left(\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}-1\right)} = \frac{W_t}{P_t} C_t^{-\frac{1}{\sigma}} Q_t^{\frac{\eta}{\sigma}-1} \\ &\Rightarrow \quad Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi H_t^{\varphi} = \frac{W_t}{P_t}\end{aligned}$$

Note that $Q_t \equiv \left(\frac{XX_t}{C_t^{\frac{\eta-1}{\eta}}}\right)^{\frac{1}{1-\eta}} \Rightarrow XX_t = \left(\frac{Q_t}{C_t^{\frac{1}{\eta}}}\right)^{1-\eta}$.

Demand for assets:

Demand for bonds:

$$
\begin{aligned}
\lambda_t &= C_t^{-\frac{1}{\eta}} P_t^{-1} XX_t^{\frac{-\frac{1}{\sigma}+\frac{1}{\eta}}{1-\frac{1}{\eta}}}, \lambda_{t+1} = C_{t+1}^{-\frac{1}{\eta}} P_{t+1}^{-1} XX_{t+1}^{\frac{-\frac{1}{\sigma}+\frac{1}{\eta}}{1-\frac{1}{\eta}}}, \lambda_t = \beta \boldsymbol{E}_t \left[\lambda_{t+1}(1+i_t^S)\right] \\
\Rightarrow & \quad 1 = \beta \boldsymbol{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t}(1+i_t^S)\right] = \beta \boldsymbol{E}_t \left[\left(\left(\frac{XX_{t+1}}{XX_t}\right)^{\frac{1}{1-\eta}}\right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\eta}} \frac{P_t}{P_{t+1}}\right](1+i_t^S) \\
&= \beta \boldsymbol{E}_t \left[\frac{C_{t+1}^{\frac{1}{\eta}-\frac{1}{\sigma}} Q_{t+1}^{\frac{\eta}{\sigma}-1}}{C_t^{\frac{1}{\eta}-\frac{1}{\sigma}} Q_t^{\frac{\eta}{\sigma}-1}} \frac{C_{t+1}^{-\frac{1}{\eta}}}{C_t^{-\frac{1}{\eta}}} \frac{P_t}{P_{t+1}}\right](1+i_t^S) \\
\Rightarrow & \quad \text{Euler Eq: } \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}}\right]\left(1+i_t^S\right) = 1 \qquad (5.132)
\end{aligned}
$$

We have used a change of variables from XX to Q:

$$
\begin{aligned}
Q_t &\equiv \left(\frac{XX_t}{C_t^{\frac{\eta-1}{\eta}}}\right)^{\frac{1}{1-\eta}} = \left(\begin{array}{c} 1+\omega_D \left(\frac{(D_t/P_t)^{1-\frac{1}{v}} + \frac{\omega_{FD}}{\omega_D}(F_t/P_t)^{1-\frac{1}{v}}}{C_t^{1-\frac{1}{v}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{v}}} \\ +\omega_E \left(\frac{(E_t/P_t)^{1-\frac{1}{\varrho}} + \frac{\omega_{FE}}{\omega_E}(F_t/P_t)^{1-\frac{1}{\varrho}}}{C_t^{1-\frac{1}{\varrho}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\varrho}}} \end{array}\right)^{\frac{1}{1-\eta}} \\
Q_t &= \left(\begin{array}{c} 1+\omega_D \left(\frac{(D_t/P_t)^{1-\frac{1}{v}} + \frac{\omega_{FD}}{\omega_D}(F_t/P_t)^{1-\frac{1}{v}}}{C_t^{1-\frac{1}{v}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{v}}} \\ +\omega_E \left(\frac{(E_t/P_t)^{1-\frac{1}{\varrho}} + \frac{\omega_{FE}}{\omega_E}(F_t/P_t)^{1-\frac{1}{\varrho}}}{C_t^{1-\frac{1}{\varrho}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\varrho}}} \end{array}\right)^{\frac{1}{1-\eta}} \\
Q_t &= \left(1+\omega_D Q_{D,t}^{1-\frac{1}{\eta}} + \omega_E Q_{E,t}^{1-\frac{1}{\eta}}\right)^{\frac{1}{1-\eta}}.
\end{aligned}
$$

Now, we derive similar equations for deposits, cash and CBDC:

$$
\begin{aligned}
1 &= \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^D + \frac{i_t^S - i_t^D}{1 + i_t^S} \left(1 + i_t^S\right) \right) \\
1 &= \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^D\right) + \frac{i_t^S - i_t^D}{1 + i_t^S} \\
1 &= \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^D\right) \\
&\quad + \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\upsilon}} \omega_D \left[\left(\frac{D_t}{P_t C_t} \right)^{1-\frac{1}{\upsilon}} + \frac{\omega_{FD}}{\omega_D} \left(\frac{F_t}{P_t C_t} \right)^{1-\frac{1}{\upsilon}} \right]^{\frac{\frac{1}{\upsilon}-\frac{1}{\eta}}{1-\frac{1}{\upsilon}}}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\text{Deposit demand:} \quad & 1 = \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^D\right) + \omega_D \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}}, \\
\text{Cash demand:} \quad & 1 = \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^E\right) + \omega_E \left(\frac{P_t C_t}{E_t} \right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}}, \\
\text{CBDC demand:} \quad & 1 = \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^F\right) \\
& + \omega_{FD} \left(\frac{P_t C_t}{F_t} \right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}} + \omega_{FE} \left(\frac{P_t C_t}{F_t} \right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}}.
\end{aligned}
$$

<u>All equilibrium conditions of the full model with cash</u>

$$
\begin{aligned}
\text{Illiquid bond demand:} \quad & \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^S\right) = 1 \\
\text{Deposit demand:} \quad & \frac{i_t^S - i_t^D}{1 + i_t^S} = \omega_D V_{D,t}^{\frac{1}{\upsilon}} V_{FD,t}^{-\frac{1}{\upsilon}+\frac{1}{\eta}} \\
\text{Cash demand:} \quad & \frac{i_t^S - i_t^E}{1 + i_t^S} = \omega_E V_{E,t}^{\frac{1}{\varrho}} V_{FE,t}^{-\frac{1}{\varrho}+\frac{1}{\eta}} \\
\text{CBDC demand:} \quad & \frac{i_t^S - i_t^F}{1 + i_t^S} = \omega_{FD} V_{F,t}^{\frac{1}{\upsilon}} V_{FD,t}^{-\frac{1}{\upsilon}+\frac{1}{\eta}} + \omega_{FE} V_{F,t}^{\frac{1}{\varrho}} V_{FE,t}^{-\frac{1}{\varrho}+\frac{1}{\eta}}
\end{aligned}
$$

$$
\text{Production: } Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}
$$

$$\text{Market clearing: } Y_t = C_t + I_t + \frac{\kappa}{2}\left(\frac{P_t}{P_{t-1}} - 1\right)^2$$

$$K_{t+1} = b_t.$$

$$K_{t+1} = I_t + (1-\delta)K_t.$$

$$\begin{aligned}
\text{Labor demand:} \quad & w_t = p_{mt}(1-\alpha)\frac{Y_t}{L_t} \\
\text{Capital demand:} \quad & 1 + r_t^K = \alpha p_{mt}\frac{Y_t}{K_t} + 1 - \delta \\
\text{Labor supply:} \quad & w_t = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi L_t^{\varphi}
\end{aligned}$$

$$\rightarrow p_{mt}Y_t = \frac{\left(\delta + r_t^K\right)K_t}{\alpha} = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi \frac{L_t^{1+\varphi}}{(1-\alpha)}$$

$$\text{Optimal pricing: } \left[\frac{\epsilon - 1}{\epsilon} - p_{mt}\right]\frac{\epsilon Y_t}{\kappa} + \frac{P_t}{P_{t-1}}\left(\frac{P_t}{P_{t-1}} - 1\right) = E_t\left[\Lambda_{t,t+1}\frac{P_{t+1}}{P_t}\left(\frac{P_{t+1}}{P_t} - 1\right)\right]$$

$$\begin{aligned}
\text{Binding leverage constraint:} \quad & D_t = \ell(M_t + \rho P_t b_t) \\
\text{Bank FOCs:} \quad & \frac{i_t^S - i_t^D}{1 + i_t^S} = \frac{i_t^S - i_t^M}{\left(1 + i_t^S\right)\ell} = \frac{i_t^S - E_t r_{t+1}^K - \left(1 + r^K\right)E_t\left(\pi_{t+1} - 1\right)}{\left(1 + i_t^S\right)\rho\ell}
\end{aligned}$$

where

$$\text{Velocity:} V_{J,t} \equiv \frac{P_t C_t}{J_t} \text{ for } J \in \{D, E, F\}$$

$$
\begin{aligned}
V_{FD,t} &\equiv \left(V_{D,t}^{-(1-\frac{1}{v})} + \frac{\omega_{FD}}{\omega_D} V_{F,t}^{-\left(1-\frac{1}{v}\right)}\right)^{\frac{-1}{1-\frac{1}{v}}} \\
V_{FE,t} &\equiv \left(V_{E,t}^{-\left(1-\frac{1}{\varrho}\right)} + \frac{\omega_{FE}}{\omega_E} V_{F,t}^{-\left(1-\frac{1}{\varrho}\right)}\right)^{\frac{-1}{1-\frac{1}{\varrho}}} \\
Q_t &\equiv \left(1 + \omega_D V_{FD,t}^{\frac{1}{\eta}-1} + \omega_E V_{FE,t}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}}
\end{aligned}
$$

Steady state equations with cash

Unknowns:

- Output, consumption and labor: Y, C, L
- Deposits, cash, CBDC and reserves balances: D, E, F, M
- Real assets: b
- Rates: i^K, i^D

Note that the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. We now derive the steady state values:

$$
\begin{aligned}
\text{Intermediate good price:} &\quad p_m = \frac{\epsilon - 1}{\epsilon}\,, \\
\text{Illiquid bond demand:} &\quad \beta\left(1 + i^S\right) = 1.
\end{aligned}
$$

The price of the intermediate good is $\frac{\epsilon-1}{\epsilon}$ in terms of the final good. This gives the markup of $\frac{1}{\epsilon-1}$, which is simply due to the market power. We solve the model in four blocks below. This solution method also reveals the transmission of monetary policy in the steady state.

Block 1: Given i^M, we can solve for i^K and i^D:

$$
\frac{i^S - i^D}{1 + i^S} = \frac{i^S - i^M}{\left(1 + i^S\right)\ell} = \frac{i^S - i^K}{\left(1 + i^S\right)\rho\ell}. \tag{5.133}
$$

Block 2: Given i^D from block 1 and $i^E = 0$ and i^F from policy, we can pin down V_D, V_E and V_F:

$$\text{Deposit demand:} \quad \frac{i^S - i^D}{1+i^S} = \omega_D V_D^{\frac{1}{\upsilon}} Q_D^{-\frac{1}{\upsilon}+\frac{1}{\eta}}, \tag{5.134}$$

$$\text{Cash demand:} \quad \frac{i^S - i^E}{1+i^S} = \omega_E V_E^{\frac{1}{\varrho}} Q_E^{-\frac{1}{\varrho}+\frac{1}{\eta}}, \tag{5.135}$$

$$\text{CBDC demand:} \quad \frac{i^S - i^F}{1+i^S} = \omega_{FD} V_F^{\frac{1}{\upsilon}} Q_D^{-\frac{1}{\upsilon}+\frac{1}{\eta}} + \omega_{FE} V_F^{\frac{1}{\varrho}} Q_E^{-\frac{1}{\varrho}+\frac{1}{\eta}}. \tag{5.136}$$

Note that Q_D and Q_E are functions of V_J's. We can then calculate Q.

Moreover, we have

$$b = K = \frac{I}{\delta}.$$

Block 3: Given i^K (from block 1) and Q (from block 2), the following four equations pin down Y, L, C and b:

$$\begin{aligned} Y &= C + \delta K, \\ p_m &= \frac{\epsilon - 1}{\epsilon} = \frac{\left(i^K + \delta\right) K}{\alpha Y} \rightarrow K = \frac{\epsilon - 1}{\epsilon} \frac{\alpha Y}{i^K + \delta}, \\ Y &= A K^{\alpha} L^{1-\alpha}, \\ \frac{\epsilon - 1}{\epsilon} Y &= Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi \frac{L^{1+\varphi}}{1-\alpha}. \end{aligned} \tag{5.137}$$

We can now calculate Y as a function of C and K and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon - 1}{\epsilon} \frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} K^{\alpha \frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi},$$

$$Y = C + \delta K \rightarrow C = Y \left(1 - \frac{\epsilon - 1}{\epsilon} \frac{\alpha \delta}{i^K + \delta}\right). \tag{5.138}$$

Given that K and C are now given in terms of Y from Equation (5.137) and Equation (5.138), we obtain

$$Y^{\varphi + \frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}} (1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} \left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}+1}}{\psi Q^{1-\frac{\eta}{\sigma}} \left(1 - \frac{\epsilon-1}{\epsilon} \frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}} \left(i^K + \delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{5.139}$$

Log-linearization around the zero-inflation-rate steady state in the full model with cash

We generally use small-case letters for log-linearized form, i.e., $\widehat{c}_t$ is log-lin of C_t.

Definition of constants:

$$
\begin{aligned}
\widetilde{x} &= \widehat{x}_t - \widehat{p}_t \text{ for } x \in \{d, e, f, m\} \\
\alpha_{FD} &\equiv \frac{\omega_{FD} V_F^{\frac{1}{\upsilon}} V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}}}{\omega_{FD} V_F^{\frac{1}{\upsilon}} V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}} + \omega_{FE} V_E^{\frac{1}{\varrho}} V_{FE}^{\frac{1}{\eta}-\frac{1}{\varrho}}} \\
\beta_J &\equiv \frac{V_D^{-(1-\frac{1}{\upsilon})}}{V_D^{-(1-\frac{1}{\upsilon})} + \frac{\omega_{FD}}{\omega_D} V_F^{-\left(1-\frac{1}{\upsilon}\right)}}, \\
\beta_E &\equiv \frac{V_E^{-(1-\frac{1}{\varrho})}}{V_E^{-(1-\frac{1}{\varrho})} + \frac{\omega_{FE}}{\omega_E} V_F^{-\left(1-\frac{1}{\varrho}\right)}}.
\end{aligned}
$$

$$
\alpha_m \equiv \frac{M/P}{M/P + \rho b}.
$$

$$
\alpha_c \equiv \frac{C}{Y}, \text{ and } \alpha_y \equiv \frac{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K}}{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K} + 1 - \delta}.
$$

$$
\alpha_{JJ} \equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_{FE}^{\frac{1}{\eta}-1}}, J \in \{D, E\}
$$

Derivations of log-linearized HHs' FOCs:

We start with HHs' FOC:

$$
\text{FOC:} \frac{i_t^S - i_t^F}{1 + i_t^S} = \omega_{FD} V_{F,t}^{\frac{1}{\upsilon}} V_{FD,t}^{\frac{1}{\eta}-\frac{1}{\upsilon}} + \omega_{FE} V_{E,t}^{\frac{1}{\varrho}} V_{FE,t}^{\frac{1}{\eta}-\frac{1}{\varrho}}
$$

The LHS:

$$\begin{aligned}
\log\left(\frac{i_t^S - i_t^F}{1+i_t^S} / \frac{i^S - i^F}{1+i^S}\right) &= \log\left(\frac{i_t^S - i_t^F}{i^S - i^F}\right) - \log\left(\frac{1+i_t^S}{1+i^S}\right) \\
\log\left(\frac{i_t^S - i_t^F}{i^S - i^F}\right) &= \log\left(1 + \frac{i_t^S - i^S - \left(i_t^F - i^F\right)}{i^S - i^F}\right) \approx \frac{i_t^S - i_t^F}{i^S - i^F} - 1 \\
\log\left(\frac{1+i_t^S}{1+i^S}\right) &= \log\left(1 + \frac{i_t^S - i^S}{1+i^S}\right) \approx \frac{1}{i^S - i^F} \frac{\left(i_t^S - i^S\right)\left(i^S - i^F\right)}{\left(1+i^S\right)} \approx 0
\end{aligned}$$

Note that the last approximation is due to the fact that $\left(i_t^S - i^S\right)\left(i^S - i^F\right)$ is a second-order term while $i_t^S - i_t^F$ is a first-order one, so we can ignore the former against the latter. Therefore,

$$\log\left(\frac{i_t^S - i_t^F}{1+i_t^S} / \frac{i^S - i^F}{1+i^S}\right) \approx \frac{i_t^S - i_t^F}{i^S - i^F} - 1.$$

The RHS:

$$\begin{aligned}
& \log\left(\frac{\omega_{FD} V_{F,t}^{\frac{1}{\upsilon}} V_{FD,t}^{\frac{1}{\eta}-\frac{1}{\upsilon}} + \omega_{FE} V_{E,t}^{\frac{1}{\varrho}} V_{FE,t}^{\frac{1}{\eta}-\frac{1}{\varrho}}}{\omega_{FD} V_{F}^{\frac{1}{\upsilon}} V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}} + \omega_{FE} V_{E}^{\frac{1}{\varrho}} V_{FE}^{\frac{1}{\eta}-\frac{1}{\varrho}}}\right) \\
\approx\; & \alpha_{FD}\left(\frac{1}{\upsilon}\widehat{V_{F,t}} + \left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)\widehat{V_{FD,t}}\right) + (1-\alpha_{FD})\left(\frac{1}{\varrho}\widehat{V_{E,t}} + \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\widehat{V_{FE,t}}\right) \\
=\; & \alpha_{FD}\left(\frac{1}{\upsilon}\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right) + \left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)\left(\beta_D\left(\widehat{p}_t + \widehat{c}_t - \widehat{d_t}\right) + (1-\beta_D)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right)\right) \\
& + (1-\alpha_{FD})\left(\frac{1}{\varrho}\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right) + \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\left(\beta_E\left(\widehat{p}_t + \widehat{c}_t - \widehat{e}_t\right) + (1-\beta_E)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right)\right) \\
=\; & \left[\begin{array}{c} \alpha_{FD}\left(\frac{1}{\upsilon} + \left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)(1-\beta_D)\right) \\ + (1-\alpha_{FD})\left(\frac{1}{\varrho} + \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)(1-\beta_E)\right) \end{array}\right]\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right) \\
& + \alpha_{FD}\left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)\beta_D\left(\widehat{p}_t + \widehat{c}_t - \widehat{d_t}\right) + (1-\alpha_{FD})\left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\beta_E\left(\widehat{p}_t + \widehat{c}_t - \widehat{e}_t\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{i_t^S - i_t^F}{i^S - i^F} - 1 &= \frac{1}{\eta}\left(\widehat{p}_t + \widehat{c}_t\right) - \alpha_{FD}\left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)\beta_D\widehat{d_t} - (1-\alpha_{FD})\left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\beta_E\widehat{e}_t \\
& \quad - \left[\alpha_{FD}\left(\frac{1-\beta_D}{\eta} + \frac{\beta_D}{\upsilon}\right) + (1-\alpha_{FD})\left(\frac{1-\beta_E}{\eta} + \frac{\beta_E}{\varrho}\right)\right]\widehat{f}_t \qquad (5.140)
\end{aligned}$$

where we have used

$$
\begin{aligned}
\alpha_{FD} &\equiv \frac{\omega_{FD} V_F^{\frac{1}{v}} V_{FD}^{\frac{1}{\eta}-\frac{1}{v}}}{\omega_{FD} V_F^{\frac{1}{v}} V_{FD}^{\frac{1}{\eta}-\frac{1}{v}} + \omega_{FE} V_E^{\frac{1}{\varrho}} V_{FE}^{\frac{1}{\eta}-\frac{1}{\varrho}}} \\
\beta_D &\equiv \frac{V_D^{-(1-\frac{1}{v})}}{V_D^{-(1-\frac{1}{v})} + \frac{\omega_{FD}}{\omega_D} V_F^{-(1-\frac{1}{v})}} \\
\beta_E &\equiv \frac{V_E^{-(1-\frac{1}{\varrho})}}{V_E^{-(1-\frac{1}{\varrho})} + \frac{\omega_{FE}}{\omega_E} V_F^{-\left(1-\frac{1}{\varrho}\right)}} \\
\widehat{V}_{J,t} &= \log\left(\frac{V_{J,t}}{V_J}\right) = \widehat{p}_t + \widehat{c}_t - \widehat{j}_t \text{ for } J \in \{D, E\} \\
\widehat{V}_{FJ,t} &= \beta_J \widehat{V}_{J,t} + (1-\beta_J)\widehat{V}_{F,t} = \beta_J\left(\widehat{p}_t + \widehat{c}_t - \widehat{j}_t\right) + (1-\beta_J)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right) \text{ for } J \in \{D, E\}
\end{aligned}
$$

Similarly, we can derive "money demand" for deposits and cash:

$$
\begin{aligned}
\frac{i_t^S - i_t^D}{i^S - i^D} - 1 &= \frac{1}{v}\widehat{V}_{D,t} + \left(\frac{1}{\eta} - \frac{1}{v}\right)\widehat{V}_{FD,t} \\
&= \frac{1}{v}\left[\widehat{p}_t + \widehat{c}_t - \widehat{d_t}\right] + \left(\frac{1}{\eta} - \frac{1}{v}\right)\left[\beta_D\left(\widehat{p}_t + \widehat{c}_t - \widehat{d_t}\right) + (1-\beta_D)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right] \rightarrow \\
\frac{i_t^S - i_t^D}{i^S - i^D} - 1 &= \frac{1}{\eta}\left(\widehat{p}_t + \widehat{c}_t\right) - \left(\frac{1-\beta_D}{v} + \frac{\beta_D}{\eta}\right)\widehat{d_t} - \left(\frac{1}{\eta} - \frac{1}{v}\right)(1-\beta_D)\widehat{f}_t
\end{aligned} \tag{5.141}
$$

$$
\begin{aligned}
\frac{i_t^S}{i^S} - 1 &= \frac{1}{\varrho}\widehat{V}_{E,t} + \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\widehat{V}_{FE,t} \\
&= \frac{1}{\varrho}\left[\widehat{p}_t + \widehat{c}_t - \widehat{e}_t\right] + \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\left[\beta_E\left(\widehat{p}_t + \widehat{c}_t - \widehat{e}_t\right) + (1-\beta_E)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right] \rightarrow \\
\frac{i_t^S}{i^S} - 1 &= \frac{1}{\eta}\left(\widehat{p}_t + \widehat{c}_t\right) - \left(\frac{1-\beta_E}{\varrho} + \frac{\beta_E}{\eta}\right)\widehat{e}_t - \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)(1-\beta_E)\widehat{f}_t
\end{aligned} \tag{5.142}
$$

$$
\begin{aligned}
Q_t &\equiv \left(1+\omega_D V_{FD,t}^{\frac{1}{\eta}-1}+\omega_E V_{FE,t}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \\
\widehat{q}_t &= \frac{1}{\eta}\left(\alpha_{DD}\widehat{V}_{FD,t}+\alpha_{EE}\widehat{V}_{FE,t}\right) \\
&= \frac{1}{\eta}\left(\alpha_{DD}\left(\beta_D\widehat{V}_{D,t}+(1-\beta_D)\widehat{V}_{F,t}\right)+\alpha_{EE}\left(\beta_E\widehat{V}_{E,t}+(1-\beta_E)\widehat{V}_{F,t}\right)\right) \\
&= \frac{1}{\eta}\left(\begin{array}{c}\alpha_{DD}\left(\beta_D\left(\widehat{p}_t+\widehat{c}_t-\widehat{d}_t\right)+(1-\beta_D)\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right)\right)\\ +\alpha_{EE}\left(\beta_E\left(\widehat{p}_t+\widehat{c}_t-\widehat{e}_t\right)+(1-\beta_E)\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right)\right)\end{array}\right) \\
&= \frac{1}{\eta}\left((\alpha_{DD}+\alpha_{EE})\left(\widehat{p}_t+\widehat{c}_t\right)-\left(\begin{array}{c}+\alpha_{DD}\beta_D\widehat{d}_t+\alpha_{EE}\beta_E\widehat{e}_t\\ +\left[\alpha_{DD}(1-\beta_D)+\alpha_{EE}(1-\beta_E)\right]\widehat{f}_t\end{array}\right)\right) \qquad \text{(QLL)}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{DD} &\equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1+\omega_D V_{FD}^{\frac{1}{\eta}-1}+\omega_E V_{FE}^{\frac{1}{\eta}-1}}, \\
\alpha_{EE} &\equiv \frac{\omega_E V_{FE}^{\frac{1}{\eta}-1}}{1+\omega_D V_{FD}^{\frac{1}{\eta}-1}+\omega_E V_{FE}^{\frac{1}{\eta}-1}}.
\end{aligned}
$$

Now we log-linearize the Euler equation for illiquid bond demand:

$$
\begin{aligned}
E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+\frac{i_t^S-i^S}{1+i^S}\right) &= 1 \\
\left(\frac{\eta}{\sigma}-1\right)\left(E_t\left[\widehat{q}_{t+1}\right]-\widehat{q}_t\right)-\frac{1}{\sigma}\left(E_t\left[\widehat{c}_{t+1}\right]-\widehat{c}_t\right)-E_t\left[\widehat{p}_{t+1}\right]+\frac{i_t^S-i^S}{1+i^S} &= 0 \\
\left(\frac{\eta}{\sigma}-1\right)\left(E_t\left[\widehat{q}_{t+1}\right]-\widehat{q}_t\right)-\frac{1}{\sigma}\left(E_t\left[\widehat{c}_{t+1}\right]-\widehat{c}_t\right)-E_t\left[\Delta\widehat{p}_{t+1}\right]+\beta i_t^S+\beta-1 &= 0
\end{aligned}
$$

$$
\widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right]-\sigma\left(\beta i_t^S-E_t\left[\Delta\widehat{p}_{t+1}\right]+\beta-1\right)+(\sigma-\eta)\left(E_t\left[\widehat{q}_{t+1}\right]-\widehat{q}_t\right)
$$

Log-lin of Philips curve:

Using $x_t \approx x(1+\widehat{x}_t)$, we can write $\Lambda_{t,t+1}\approx\beta(1+\widehat{\Lambda}_{t+1})$ and $\Lambda_{t,t+1}\equiv\beta\frac{U_{c,t+1}}{U_{c,t}}=\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\eta}}$, so $\widehat{\Lambda}_{t+1}=$

$-\frac{1}{\eta}\left(\widehat{c}_{t+1}-\widehat{c}_t\right).$

$$E_t\left[\Lambda_{t,t+1}\pi_{t+1}\right]=E_t\left[\Lambda_{t,t+1}\pi_{t+1}\right]\approx E_t\left[\beta(1-\frac{1}{\eta}\left(\widehat{c}_{t+1}-\widehat{c}_t\right))\pi_{t+1}\right]\approx E_t\left[\beta\pi_{t+1}\right]$$

We have

$$\begin{aligned}\left[\frac{\epsilon-1}{\epsilon}-p_{mt}\right]\frac{\epsilon Y_t}{\kappa}+\frac{P_t}{P_{t-1}}\left(\frac{P_t}{P_{t-1}}-1\right) &= E_t\left[\Lambda_{t,t+1}\frac{P_{t+1}}{P_t}\left(\frac{P_{t+1}}{P_t}-1\right)\right]\\ \left[\frac{\epsilon-1}{\epsilon}-p_m(1+\widehat{p}_{mt})\right]\epsilon Y\left(1+\widehat{y}_t\right)+\kappa\left(1+\widehat{\pi}_t\right)\widehat{\pi}_t &\approx \kappa E_t\left[\Lambda_{t,t+1}\left(1+\widehat{\pi}_{t+1}\right)\widehat{\pi}_{t+1}\right]\\ \widehat{\pi}_t &\approx \frac{(\epsilon-1)Y}{\kappa}\widehat{p}_{mt}+\beta E_t\left[\widehat{\pi}_{t+1}\right] \qquad (5.143)\end{aligned}$$

where we used the following notation: $\widehat{\pi}_t\equiv\Delta\hat{p}_t\equiv\log\frac{P_t}{P_{t-1}}$.

Rest of equations:

$$Y_t=A_tK_t^{\alpha}L_t^{1-\alpha}\rightarrow\widehat{y}_t=\alpha\widehat{k}_t+(1-\alpha)\,\widehat{l}_t$$

$$Y_t=C_t+I_t+\frac{\kappa}{2}\left(\frac{P_t}{P_{t-1}}-1\right)^2\rightarrow\widehat{y}_t=\alpha_c\widehat{c}_t+(1-\alpha_c)\,\widehat{i}_t \qquad (5.144)$$

$$\widehat{k}_{t+1}=\widehat{b}_t$$

$$\mathbf{K}_t=I_t+(1-\delta)K_t\rightarrow\widehat{k}_{t+1}=\delta\widehat{i}_t+(1-\delta)\,\widehat{k}_t$$

$$
\begin{aligned}
w_t &= p_{mt}(1-\alpha)\frac{Y_t}{L_t} \rightarrow \widehat{w}_t = \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t \\
1 + r_t^K &= \alpha p_{mt}\frac{Y_t}{K_t} + 1 - \delta \rightarrow \frac{r_t^K - r^K}{1 + r^K} = \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) \qquad (5.145) \\
w_t &= Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi H_t^{\varphi} \\
\widehat{w}_t &= \left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi\widehat{l}_t \\
&= \frac{\sigma - \eta}{\sigma\eta}\left((\alpha_{DD} + \alpha_{EE})\left(\widehat{p}_t + \widehat{c}_t\right) - \begin{pmatrix} +\alpha_{DD}\beta_D\widehat{d}_t + \alpha_{EE}\beta_E\widehat{e}_t \\ +\left[\alpha_{DD}(1-\beta_D) + \alpha_{EE}(1-\beta_E)\right]\widehat{f}_t \end{pmatrix}\right) + \frac{1}{\sigma}\widehat{c}_t + \varphi\widehat{l}_t \qquad (5.146)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_c &\equiv \frac{C}{Y} \\
\alpha_y &\equiv \frac{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K}}{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K} + 1 - \delta}
\end{aligned}
$$

Note that $r^K = i^K$ because inflation is zero in the SS.

We have also used labor market clearing condition $H_t = L_t$.

Summary of log-linearized equations with cash:

Define $\widetilde{x} = \widehat{x}_t - \widehat{p}_t$ for $x \in \{d, e, f, m\}$.

Parameters α_{FD}, β_J, α_m, α_c, α_y and α_{JJ} for $J \in \{D, E\}$ are all constants and defined in the appendix. Also define $\widehat{\pi}_t \equiv \Delta\hat{p}_t$. We have the following log-lin equations.

Here is the summary of log-linearized version of equilibrium conditions:

$$
\text{Euler equation: } \widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right] - \sigma\left(\beta i_t^S - E_t\left[\widehat{\pi}_{t+1}\right] + \beta - 1\right) + (\sigma - \eta)\left(E_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \qquad (5.147)
$$

$$
\begin{aligned}
\text{CBDC demand:} \quad & \tfrac{i_t^S - i_t^F}{i^S - i^F} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \alpha_{FD}\left(\tfrac{1}{\eta} - \tfrac{1}{\upsilon}\right)\beta_D\widetilde{d} - (1-\alpha_{FD})\left(\tfrac{1}{\eta} - \tfrac{1}{\varrho}\right)\beta_E\widetilde{e} \\
& \quad - \left[\alpha_{FD}\left(\tfrac{1-\beta_D}{\eta} + \tfrac{\beta_D}{\upsilon}\right) + (1-\alpha_{FD})\left(\tfrac{1-\beta_E}{\eta} + \tfrac{\beta_E}{\varrho}\right)\right]\widetilde{f} \qquad (5.148) \\
\text{Deposit demand:} \quad & \tfrac{i_t^S - i_t^D}{i^S - i^D} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \left(\tfrac{1-\beta_D}{\upsilon} + \tfrac{\beta_D}{\eta}\right)\widetilde{d} - \left(-\tfrac{1}{\upsilon} + \tfrac{1}{\eta}\right)(1-\beta_D)\widetilde{f} \qquad (5.149) \\
\text{Cash demand:} \quad & \tfrac{i_t^S}{i^S} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \left(\tfrac{1-\beta_E}{\varrho} + \tfrac{\beta_E}{\eta}\right)\widetilde{e} - \left(-\tfrac{1}{\varrho} + \tfrac{1}{\eta}\right)(1-\beta_E)\widetilde{f} \qquad (5.150)
\end{aligned}
$$

Bank equations:

$$i_t^S - i_t^D = \ell^{-1}\left(i_t^S - i_t^M\right) \tag{5.151}$$

$$i_t^S - E_t r_{t+1}^K - \left(1 + r^K\right) E_t\left[\widehat{\pi}_{t+1}\right] = \rho\left(i_t^S - i_t^M\right) \tag{5.152}$$

$$\widetilde{d} = \alpha_m \widetilde{m} + (1 - \alpha_m)\,\widehat{b}_t \tag{5.153}$$

Philips curve:

$$\widehat{\pi}_t = \frac{(\epsilon - 1)\,Y}{\kappa}\widehat{p}_{mt} + \beta E_t\left[\widehat{\pi}_{t+1}\right] \tag{5.154}$$

The rest of the equations:

$$\widehat{y}_t = \alpha \widehat{k}_t + (1 - \alpha)\,\widehat{l}_t \tag{5.155}$$

$$\widehat{y}_t = \alpha_c \widehat{c}_t + (1 - \alpha_c)\,\widehat{i}_t \tag{5.156}$$

$$\widehat{k}_{t+1} = \widehat{b}_t \tag{5.157}$$

$$\widehat{k}_{t+1} = \delta \widehat{i}_t + (1 - \delta)\,\widehat{k}_t + \widehat{\xi}_{t+1} \tag{5.158}$$

$$\widehat{w}_t = \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t \tag{5.159}$$

$$\frac{r_t^K - r^K}{1 + r^K} = \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) + \widehat{\xi}_{t+1} \tag{5.160}$$

$$\widehat{w}_t = \left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi \widehat{l}_t \tag{5.161}$$

$$\widehat{q}_t = \frac{1}{\eta}\left(\begin{array}{c} (\alpha_{DD} + \alpha_{EE})\,\widehat{c}_t \\ -\left(\begin{array}{c} +\alpha_{DD}\beta_D \widetilde{d} + \alpha_{EE}\beta_E \widetilde{e} \\ +\left[\alpha_{DD}(1 - \beta_D) + \alpha_{EE}(1 - \beta_E)\right]\widetilde{f}_t \end{array}\right) \end{array}\right) \tag{5.162}$$

G Special cases of the model

Now consider the general model with cash. Within this general model, we discuss some special cases that have been studied in the literature before: the no-CBDC case, a cash-like CBDC and a deposit-like CBDC. The goal is to show how this model can easily nest other models in the literature.

Special case: No CBDC

Here, we assume $\omega_{FD} = \omega_{FE} = 0$. The optimality conditions are modified to

$$\begin{aligned} \text{Deposits} \quad &: \quad \frac{i^S - i^D}{1+i^S} = \omega_D \left(\frac{P_t C_t}{D_t}\right)^{\frac{1}{\eta}}, \\ \text{Cash} \quad &: \quad \frac{i^S}{1+i^S} = \omega_E \left(\frac{P_t C_t}{E_t}\right)^{\frac{1}{\eta}}, \end{aligned}$$

where

$$\begin{aligned} Q_{D,t} &\equiv V_{D,t}^{-1} = \frac{D_t}{P_t C_t}, Q_{E,t} \equiv V_{E,t}^{-1} = \frac{E_t}{P_t C_t}, \\ Q_t &\equiv \left(1 + \omega_D V_{D,t}^{-(1-\frac{1}{\eta})} + \omega_E V_{E,t}^{-(1-\frac{1}{\eta})}\right)^{\frac{1}{1-\eta}}. \end{aligned}$$

This case nests Piazzesi et al.'s (2019) description of households. However, our description is still more general because not only deposits but also cash provides liquidity services here.

Special case: $\varrho = \upsilon = \eta$

In this case, agents have the love-of-variety feature in their means of payments, and the elasticity of consumption with respect to different means of payments are identical. The introduction of a CBDC here just enriches the set of means of payments that agents have available. The optimal conditions here imply that

$$\frac{i^S - i^D}{1+i^S} \geq \omega_D \left(\frac{P_t C_t}{D_t}\right)^{\frac{1}{\eta}} \text{ with equality if } D_t > 0 \tag{5.163}$$

$$\frac{i^S}{1+i^S} \geq \omega_E \left(\frac{P_t C_t}{E_t}\right)^{\frac{1}{\eta}} \text{ with equality if } E_t > 0 \tag{5.164}$$

$$\frac{i^S - i^F}{1+i^S} \geq (\omega_{FD} + \omega_{FE}) \left(\frac{P_t C_t}{F_t}\right)^{\frac{1}{\eta}} \text{ with equality if } F_t > 0 \tag{5.165}$$

Here, the demand for different means of payments are not inter-related. The opportunity cost of each means of payment pins down the velocity and demand for that. This is true even if the utility function is not separable. The effect of non-separability will be reflected in the labor supply equation.

CBDC is a perfect substitute for deposits: $\upsilon = \infty$

Here, we study a CBDC that is a perfect substitute for bank deposits, i.e, $\upsilon = \infty$. It is similar to a deposit-like CBDC which has been discussed in the literature, but also offers some degree of substitution with cash.[17] In this case, $Q_{D,t}$ is modified to $Q_{D,t} = V_{D,t}^{-1} + \frac{\omega_{FD}}{\omega_D} V_{F,t}^{-1} = \frac{D_t + \frac{\omega_{FD}}{\omega_D} F_t}{P_t C_t}$. Therefore, the optimality conditions imply

$$\text{Deposits:} \qquad \frac{i^S - i^D}{1+i^S} = \omega_D \left(\frac{P_t C_t}{D_t + \frac{\omega_{FD}}{\omega_D} F_t} \right)^{\frac{1}{\eta}}$$

$$\text{Cash:} \qquad \frac{i^S - i^F}{1+i^S} = \omega_{FD} \left(\frac{P_t C_t}{D_t + \frac{\omega_{FD}}{\omega_D} F_t} \right)^{\frac{1}{\eta}} + \omega_{FE} \left(\frac{P_t C_t}{F_t} \right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho} - \frac{1}{\eta}}$$

assuming that the CBDC is used in equilibrium. One can divide the second optimality condition by the first to obtain an equation for the opportunity cost of holding CBDC relative to that of deposits. For simplicity, assume $\varrho = \eta$, then we have

$$\frac{i^D - i^F}{i^S - i^D} = \frac{\omega_{FD} - \omega_D}{\omega_D} + \frac{\omega_{FE}}{\omega_D} \left(\frac{D_t}{F_t} + \frac{\omega_{FD}}{\omega_D} \right)^{\frac{1}{\eta}}.$$

This equation implies that the wedge between the interest rate of this type of CBDC and deposits depends on the relative usefulness of CBDC in deposit transactions as well as the liquidity service it provides in cash transactions. This equation is especially useful because it is a function of the relative quantity of deposits and CBDC and does not depend on the quantity of cash used in transactions (which is a direct implication of $\varrho = \eta$).

Finally, in a special case where CBDC can be used in exactly the same set of transactions as deposits with the same importance, $\omega_{FD} = \omega_D$, we will have $i^D \geq i^F$. The central bank pays less interest on CBDC compared with bank deposits as long as CBDC provides liquidity not only to deposit transactions but also in some transactions where cash can currently be used.

CBDC is a perfect substitute for cash: $\varrho = \infty$

Here we study a CBDC that is a perfect substitute for cash, i.e, $\varrho = \infty$. It is similar to the cash-like CBDC discussed in the literature but also offers some degree of substitution with deposits. In

[17] A solely deposit-like CBDC would require $\omega_{FE} = 0$.

this case, the optimality conditions imply:

$$\text{Cash:} \quad \frac{i^S}{1+i^S} \geq \omega_E Q_{E,t}^{-\frac{1}{\eta}} \text{ with eq if } E_t > 0,$$

$$\text{CBDC:} \quad \frac{i^S - i^F}{1+i^S} \geq \omega_{FD} \left(\frac{P_t C_t}{F_t}\right)^{\frac{1}{\nu}} Q_{D,t}^{\frac{1}{\nu}-\frac{1}{\eta}} + \omega_{FE} Q_{E,t}^{-\frac{1}{\eta}} \text{ with eq if } F_t > 0.$$

If cash and CBDC are both used in equilibrium, then:

$$\frac{-i^F}{1+i^S} = (\omega_{FE} - \omega_E)\, Q_{E,t}^{-\frac{1}{\eta}} + \omega_{FD} \left(\frac{P_t C_t}{F_t}\right)^{\frac{1}{\nu}} Q_{D,t}^{\frac{1}{\nu}-\frac{1}{\eta}}.$$

Note, for example, that when $\omega_{FE} = \omega_E$, a positive interest on CBDC ($i^F \geq 0$) implies that cash will be out of circulation as it is strictly dominated by CBDC. In general, the higher the difference $\omega_{FE} - \omega_E$ or the higher the usefulness of CBDC in other transactions (higher ω_{FD}), the lower the CBDC rate can go.

Now we analyze the **steady state of various special cases**.

The zero inflation rate steady state with an endogenous quantity of reserves

Special case when $\eta = \sigma = 1$. As i^M goes up, i^K goes up too.

We first assume $\frac{\eta}{\sigma} = 1$ and $\delta = 0$, for simplicity. An increase in i^M **makes loans more expensive for firms, so output goes down**. However, notice that because $\frac{\eta}{\sigma} = 1$, the payment side does not have any effects on the opportunity cost of lending and on deposit rates because they are both determined only by cost of reserves.

Notice that in this case, if CBDC and deposits are substitutes to some extent, as i^F goes down, then D goes down too, and at some point, D/P goes less than $\ell \rho b$, at which point $M \geq 0$ will be binding. This means that banks cannot attract enough deposits to raise resources for their lending.

Altogether, when $\frac{\eta}{\sigma} = 1$, the disintermediation channel does not operate in this model and the lending side is separated from the deposit side given that the reserve requirement is binding and the interest on CBDC is low enough.

Result: When $\sigma = \eta$, the output depends only on interest on reserves, i^M (and not on the interest on CBDC i^F), and i^F only determines the quantity of real balances demanded.

Special case when $\varrho = \upsilon = \eta$. In this special case, we can calculate Q in a closed form. Rewrite FOCs:

$$\text{Deposit demand:} \quad \frac{i^S - i^D}{1+i^S} = \omega_D V_D^{\frac{1}{\eta}} \tag{5.166}$$

$$\text{Cash demand:} \quad \frac{i^S - i^E}{1+i^S} = \omega_E V_E^{\frac{1}{\eta}} \tag{5.167}$$

$$\text{CBDC demand:} \quad \frac{i^S - i^F}{1+i^S} = \omega_F V_F^{\frac{1}{\eta}} \tag{5.168}$$

where $\omega_F \equiv \omega_{FD} + \omega_{FE}$. Hence, Q is given by

$$\begin{aligned} Q &\equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_{FE}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \\ &= \left(1 + \omega_D V_D^{\frac{1}{\eta}-1} + \omega_E V_E^{\frac{1}{\eta}-1} + \omega_F V_F^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \\ &= \left(1 + \sum_J \omega_J^{\eta} \left(\frac{i^S - i^J}{1+i^S}\right)^{1-\eta}\right)^{\frac{1}{1-\eta}} \end{aligned} \tag{5.169}$$

Result:

- When $\sigma \neq \eta$, the output depends on policy from two channels. First, Y is a function of i^K directly and i^K is determined by i^M. Second, Y is a function of Q, which depends on the CBDC rate (as a decreasing function) and also depends on i^D, which is again determined by i^M.

  More specifically, when $\sigma > \eta$, there is complementarity between consumption and other means of payments. Paying more interest on CBDC decreases Q and increases Y.

  Note that in this case ($\varrho = \upsilon = \eta$), we do not see the disintermediation channel because CBDC is not a perfect substitute for deposits.

Another special case: CBDC and deposits are perfect substitutes: $\upsilon = \infty$, $\omega_D = \omega_{FD}$ **and** $\omega_{FE} = 0$

$$\text{Deposit and CBDC demand:} \quad \frac{i^S - i^D}{1+i^S} = \omega_D V_{FD}^{\frac{1}{\eta}} = \frac{i^S - i^F}{1+i^S} \tag{5.170}$$

$$\text{Cash demand:} \quad \frac{i^S - i^E}{1+i^S} = \omega_E V_E^{\frac{1}{\varrho}} V_{FE}^{-\frac{1}{\varrho}+\frac{1}{\eta}} \tag{5.171}$$

$$
\begin{aligned}
V_{FD} &\equiv \left(V_D^{-1} + V_F^{-1}\right)^{-1} = \frac{PC}{D+F} \\
V_{FE} &\equiv V_E \\
Q &\equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_E^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}}
\end{aligned}
$$

In this case, interest on deposits is pinned down by i^M, on the one hand. On the other hand, it is determined by the CBDC interest rate.

More specifically, for a given i^M:

- If $i^F > i^D$, bank cannot raise deposits. Therefore, $i^D \geq i^F$.
- If $i^F < i^D$, demand for CBDC is zero and CBDC is not used.
- If $i^D = i^F$, then agents are indifferent between CBDC and deposits.

For a given i^F, the maximum deposit demand is given by

$$
\frac{PC}{V_{FD}} = \frac{PC}{\left(\frac{i^S - i^F}{\omega_D(1+i^S)}\right)^{\eta}}.
$$

- If i^M implies an i^D strictly lower than i^F, then banks cannot raise deposits, implying that production cannot take place, which is not possible. This means our initial assumption that banks hold reserves is violated. In this case, banks increase their rate to i^F to compete with CBDC and be able to raise deposits. They don't invest in reserves because their rate is too low. (They would have borrowed reserves if we had allowed them). In this case, $\frac{i^S - i^D}{1+i^S} = \frac{i^S - i^F}{1+i^S} = \frac{i^S - i^K}{(1+i^S)\rho\ell} < \frac{i^S - i^M}{(1+i^S)\ell}$. The CBDC interest rate is the floor for deposit rates. (This case is close to Keister and Sanches (2023)).
- In the knife-edge case, i^M implies an i^D exactly equal to i^F. In this case, agents are indifferent between the CBDC and deposits.
- If i^M implies an i^D higher than i^F, then the CBDC is not used.

The zero inflation rate steady state with a fixed quantity of reserves

The only difference is that here i^M is endogenous and M/P (real supply of reserves) is exogenous and set by the policy.

Endogenous variables:

- Price level, output and consumption and labor: Y, C, L
- Nominal balances: D, E
- Real assets: b
- Rates: i^M, i^K, i^D

Policy tools: $M/P, i^F$.

We follow the same three steps as before. However, since we do not know i^M, we have to start with a guess for i^M and then solve for a fixed point. More specifically:

- Start with a guess for i^M and derive the value for M/P (demand for reserves) by solving the three blocks in the previous subsection. That is, for the given i^M, start from Block 1 and follow the same steps summarized in Equations (5.1) to (5.5) and derive M/P from (5.5). Call it m_0, which is the demand for reserves.
- If m_0 is higher than the exogenous M/P set by the policy, we have to decrease i^M in the next iteration; otherwise, we have to increase i^M.
- Continue this until we converge and get the value for i^M.

Note that a higher i^M means that banks receive more benefits for reserves, so their demand for reserves goes up.